Journal of Statistical Physics, Vol. 110, Nos. 1/2, January 2003 (© 2003)

Self-Similar Asymptotics for the Boltzmann Equation with Inelastic and Elastic Interactions

A. V. Bobylev[1] **and C. Cercignani**[2]

Received March 18, 2002; accepted July 15, 2002

We consider some questions related to the self-similar asymptotics in the kinetic theory of both elastic and inelastic particles. In the second case we have in mind granular materials, when the model of hard spheres with inelastic collisions is replaced by a Maxwell model, characterized by a collision frequency independent of the relative speed of the colliding particles. We first discuss how to define the n-dimensional ($n = 1, 2,...$) inelastic Maxwell model and its connection with the more basic Boltzmann equation for inelastic hard spheres. Then we consider both elastic and inelastic Maxwell models from a unified viewpoint. We prove the existence of (positive in the inelastic case) self-similar solutions with finite energy and investigate their role in large time asymptotics. It is proved that a recent conjecture by Ernst and Brito devoted to high energy tails for inelastic Maxwell particles is true for a certain class of initial data which includes Maxwellians. We also prove that the self-similar asymptotics for high energies is typical for some classes of solutions of the classical (elastic) Boltzmann equation for Maxwell molecules. New classes of (not necessarily positive) finite-energy eternal solutions of this equation are also studied.

KEY WORDS: Granular material; Boltzmann equation; self-similar solutions.

1. INTRODUCTION

In this paper we continue a mathematically rigorous investigation of self-similar solutions of the Boltzmann equation, started in refs. 1 and 2. Our goal is to extend the results of these papers to the case of solutions with finite energy and to inelastic Maxwell models (in particular to probe the "Ernst–Brito conjecture"[3, 4]).

[1] Division of Engineering Sciences, Physics and Mathematics, Karlstad University, Karlstad, Sweden.

[2] Dipartimento di Matematica, Politecnico di Milano, Milano, Italy; e-mail: carcer@mate.polimi.it

In order to explain the idea of inelastic Maxwell models[3–10, 12, 13] we recall that the simplest model for soft granular materials is a "gas" of hard spheres, characterized by a certain restitution coefficient $0 < e < 1$. In the rarefied case this model is described by the (inelastic) Boltzmann equation. Yet, this equation is still rather complicated and we cannot use very simple models of the BGK type[11] in order to understand the qualitative behavior of the solutions. This was one of the main reasons for introducing a Maxwell model[7] by an analogy with the classical (pseudo)-Maxwell molecules, characterized by a collision frequency independent of the relative speed of colliding particles. It appears that the one-dimensional version of the model was independently introduced in ref. 12. Several interesting problems, such as the shear flow in a slab,[9] the form of the moment equations[10] and steady solutions in a thermal bath[8, 10] were already studied in the framework of the 3d Maxwell model. On the other hand, some very recent numerical experiments with 2d models and exact solutions for the 1d model (see ref. 3 for a short survey) originate many interesting questions which call for a clarification. The first, relatively simple question is:

(1) How do we define the n-dimensional ($n = 1, 2, \ldots$) inelastic Maxwell model and what connection does it have with the more basic Boltzmann equation for inelastic hard spheres?

We address this question in Section 2 written at the rather formal ("physical") level of rigor. This section is included also to prevent a further propagation (see, for example, ref. 4) of an erroneous strong form of the inelastic Maxwell model given in ref. 7 (one of the authors (A.V.B.) must apologize for it) and corrected later.[13] We show in Section 2 that the whole derivation of models can be done directly in a weak form and therefore all previously published results based on the Fourier transform remain valid. It should be mentioned that another way of introducing n-dimensional inelastic Maxwell models was considered by Ernst and Brito,[4] who, however "have not been able to perform the Fourier transform ... for general dimensionality."

The second question is:

(2) What can be said about self-similar solutions and their role in large time asymptotics for initial data possessing finite moments of any order?

We address this question in Sections 3–8 for both elastic and inelastic interactions (we refer below to these two cases as EBE and IBE, respectively). Such asymptotics was recently proved by the authors of ref. 2 for solutions of EBE with infinite energy. Independently, it was conjectured recently by Ernst and Brito[3, 4] (see also refs. 5 and 6) that a similar property holds also for solutions of IBE with finite energy (the case of finite energy is certainly more interesting from a physical viewpoint despite the fact that inelastic Maxwell models should be considered just as a rough approximation to inelastic hard spheres). In their very interesting paper[3] Ernst and Brito presented two main arguments to explain the reasons in favor of their conjecture: (a) higher moments of "rescaled" solutions of IBE tend to infinity for long times and (b) the whole family of self-similar solutions must be two-parametric (a fact well-known for EBE, see, e.g., ref. 14) having in the general case power-like tails with the "same" (see (a)) number of finite moments.

They, however, did not present any method for "extracting" this special self-similar solution satisfying their conjecture (it must be unique for a given class of initial conditions) from the whole two-parameter family. In any case their conjecture seems to be correct; we prove rigorously that (a) such unique self-similar solution does exist and (b) the conjecture is true at least for a wide class of isotropic initial conditions including Maxwellians (Sections 5–7). There is a subtle point to be understood: the tails are "high-energy" in a special sense: the time-evolution cools a granular gas to a time dependent temperature $T = T(t)$. The conjecture to be proved says that, when $t \to \infty$ (and $T(t) \to 0$), the power-like behavior is exhibited at energies which are relatively large with respect to $T(t)$, but these energies may be very low with respect to $T(0)$. On the other hand, any moment of the solution remains bounded for all $t > 0$ provided it was finite at $t = 0$. Thus, the power-like behavior at intermediate energies $T(t) \leqslant E \leqslant T(0)$, $t \to \infty$, is not very important for high energy tails.

The paper is organized as follows. After discussing inelastic Maxwell models in Section 2, we return to the classical (elastic) case and explain a formal structure of corresponding self-similar solutions with finite energy (Section 3). In the same section we establish a class of solutions which might have a self-similar asymptotics. These considerations are extended to the inelastic case in Section 4. In the same section we introduce a generalized kinetic equation (in the Fourier representation) which allows treating both the elastic and the inelastic cases from a unified viewpoint. In Section 5 we study an integral equation for (bounded!) self-similar solutions and construct its non-trivial solution f (Theorem 5.3) which appears to be unique under certain natural restrictions (uniqueness is proved at the end of Section 6). In Section 6 we prove that the self-similar solution constructed in Theorem 5.3 does represent long time asymptotics for a certain class of initial conditions (Theorem 6.2). In Section 7 we apply the results (obtained for the generalized kinetic equation) to general multi-dimensional (isotropic) Maxwell models and prove, in particular, the positivity of self-similar solutions and a weakened form (not for "all" initial conditions) of the Ernst–Brito conjecture (Theorem 7.1). Then we discuss this conjecture and our results in more detail in Section 8. Section 9 is devoted to applications of the results obtained in Sections 5 and 6 to high energy tails for the elastic Boltzmann equation. It is proved (Theorem 9.2) that the self-similar asymptotics (for high energies) is also typical for a certain class of solutions of EBE. The main results of the paper are formulated in Theorems 5.3, 6.2, 7.1, 9.2, and in Proposition 8.2. The latter relates to new classes of self-similar eternal solutions of the classical (elastic) Boltzmann equation for Maxwell molecules. For brevity we do not describe in this paper such "routine" procedures as the generalization of our results from "pseudo-" to "true" Maxwell molecules (in the elastic case) and the proof that the self-similar solutions constructed in Section 5 (Theorem 5.3) as solutions of a suitable integral equation do satisfy the initial integro-differential equation.

For the readers who are not interested in the mathematical details, we present a rough description of our main results in Section 10.

2. THE KINETIC EQUATION AND ITS FOURIER REPRESENTATION

Let $f(\mathbf{v}, t)$ be the one-particle distribution function (here $\mathbf{v} \in \mathbb{R}^n$ $(n = 2, 3,\ldots)$ and $t \in \mathbb{R}_+$ denote the velocity and time variables, respectively) of a spatially homogeneous system of inelastic particles. Then the spatially homogeneous Inelastic Boltzmann Equation (IBE) reads as follows

$$\frac{\partial f}{\partial t} = \frac{1}{2}\, d^{n-1} \int_{\mathbb{R}^n} \int_{S^{n-1}} |\mathbf{u} \cdot \mathbf{n}| \left[\frac{1}{e^2} f(t, \mathbf{v}_*)\, f(t, \mathbf{w}_*) - f(t, \mathbf{v})\, f(t, \mathbf{w}) \right] d\mathbf{n}\, d\mathbf{w}, \tag{2.1}$$

where $\mathbf{u} = \mathbf{v} - \mathbf{w}$, and $\mathbf{v}_*$, $\mathbf{w}_*$ are the pre-collisional velocities given by the following equalities

$$\mathbf{v}_* = \mathbf{v} - \frac{1+e}{2e}\,(\mathbf{u} \cdot \mathbf{n})\,\mathbf{n}, \tag{2.2}$$

$$\mathbf{w}_* = \mathbf{w} + \frac{1+e}{2e}\,(\mathbf{u} \cdot \mathbf{n})\,\mathbf{n}. \tag{2.3}$$

Then the weak form of the equation reads as follows:

$$\frac{\partial}{\partial t}(f, g) = d^{n-1} \frac{1}{4} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \int_{S^{n-1}} |\mathbf{u} \cdot \mathbf{n}|\, f(t, \mathbf{v})\, f(t, \mathbf{w}) \times [g(\mathbf{v}') + g(\mathbf{w}') - g(\mathbf{v}) - g(\mathbf{w})]\, d\mathbf{n}\, d\mathbf{v}\, d\mathbf{w}, \tag{2.4}$$

where

$$(f, g) = \int_{\mathbb{R}^n} f(t, \mathbf{v})\, g(\mathbf{v})\, d\mathbf{v}. \tag{2.5}$$

Here $g(\mathbf{v})$ is any "good" test function and $\mathbf{v}'$, $\mathbf{w}'$ are post-collisional velocities given by

$$\mathbf{v}' = \tfrac{1}{2}(\mathbf{v}+\mathbf{w}+\mathbf{u}'), \qquad \mathbf{w}' = \tfrac{1}{2}(\mathbf{v}+\mathbf{w}-\mathbf{u}'), \tag{2.6}$$

$$\mathbf{u}' = \mathbf{u}-(1+e)(\mathbf{u}\cdot\mathbf{n})\,\mathbf{n}, \qquad \mathbf{u} = \mathbf{v}-\mathbf{w}, \qquad \mathbf{u}' = \mathbf{v}'-\mathbf{w}'. \tag{2.7}$$

where $0 < e \leqslant 1$ is the restitution coefficient. We consider the case $e = \text{const.}$ only.

Without loss of generality we can assume that

$$(f, 1) = 1, \qquad (f, \mathbf{v}) = 0. \tag{2.8}$$

We also introduce the usual notation for granular temperature

$$T(t) = \frac{1}{n}(f, |\mathbf{v}|^2). \tag{2.9}$$

Following refs. 7 and 8 we can also model the system by a pseudo-Maxwellian kinetic equation. We change the factor $|\mathbf{u}\cdot\mathbf{n}|$ under the integral sign in Eq. (2.4) to $\langle|\mathbf{u}|\rangle\, |\mathbf{u}\cdot\mathbf{n}|/|\mathbf{u}|$, where the average $\langle|\mathbf{u}|\rangle$ is a function of time t only. A reasonable approximation is

$$\langle|\mathbf{u}|\rangle \cong \gamma_n \sqrt{T(t)}, \qquad \gamma_n = \text{const.} \tag{2.10}$$

where γ_n can be different for different problems. Thus we replace Eq. (2.4) by the Maxwell model for the IBE in weak form, i.e.:

$$\frac{\partial}{\partial t}(f, g) = \frac{d^n}{2}\gamma_n \sqrt{T}\,(g, Q_M(f, f)), \tag{2.11}$$

where the term in the right-hand side describes the inelastic collisions between particles. The explicit form of the of the weak form of pseudo-Maxwellian collision integral is:[7,10]

$$(g, Q_M(f, f)) = \int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\int_{S^{n-1}} f(t, \mathbf{v})\, f(t, \mathbf{w}) \left|\frac{\mathbf{u}\cdot\mathbf{n}}{|\mathbf{u}|}\right| [g(\mathbf{v}')-g(\mathbf{v})]\, d\mathbf{n}\, d\mathbf{v}\, d\mathbf{w}. \tag{2.12}$$

The main idea justifying the use of this model is that it is very advantageous if we pass to the Fourier representation by taking $g = e^{-i\mathbf{k}\cdot\mathbf{v}}$ in (2.11). Following refs. 7, 10, and 16 we introduce the characteristic function

$$\phi(\mathbf{k}, t) = \int_{\mathbb{R}^n} f(\mathbf{v}, t)\, e^{-i\mathbf{k}\cdot\mathbf{v}}\, d\mathbf{v}. \tag{2.13}$$

Then the equation for $\phi(\mathbf{k}, t)$ reads as follows (see ref. 7 for details)

$$\frac{\partial \phi}{\partial t} = d^{n-1} \gamma_n \sqrt{T}\, \mathfrak{I}(\phi, \phi), \tag{2.14}$$

where

$$\mathfrak{I}(\phi, \phi) = \frac{1}{2} \int_{S^{n-1}} d\mathbf{n} \left| \frac{\mathbf{k}\cdot\mathbf{n}}{|\mathbf{k}|} \right| [\phi(\mathbf{k}_+)\, \phi(\mathbf{k}-\mathbf{k}_+) - \phi(0)\, \phi(\mathbf{k})], \tag{2.15}$$

$$\mathbf{k}_+ = z(\mathbf{k}\cdot\mathbf{n})\, \mathbf{n}, \qquad z = \frac{1+e}{2}. \tag{2.16}$$

By re-scaling the time variable

$$\tilde{t} = d^{n-1} \gamma_n \int_0^t \sqrt{T(t')}\, dt' \tag{2.17}$$

we finally obtain the resulting n-dimensional Maxwell model in the Fourier representation (tilde is omitted)

$$\frac{\partial \phi}{\partial t} = \mathfrak{I}(\phi, \phi) = \int_{S^{n-1}} d\mathbf{n} \left(\frac{\mathbf{k}\cdot\mathbf{n}}{|\mathbf{k}|} \right)_+ [\phi(\mathbf{k}_+)\, \phi(\mathbf{k}-\mathbf{k}_+) - \phi(0)\, \phi(\mathbf{k})],$$
$$\mathbf{k} \in \mathbb{R}^n, \quad n = 2, 3, \ldots \tag{2.18}$$

This equation can be formally extended to the case $n = 1$

$$\frac{\partial \phi}{\partial t} = \mathfrak{I}(\phi, \phi) = \phi(zk)\, \phi[(1-z)\, k] - \phi(0)\, \phi(k), \qquad k \in \mathbb{R}, \tag{2.19}$$

since

$$\int_{S^{n-1}} d\mathbf{n}\, F(\mathbf{n}) = 2 \int_{\mathbb{R}^n} d\mathbf{r}\, \delta(|\mathbf{r}|^2 - 1)\, F(\mathbf{r}).$$

If $\phi(\mathbf{k}, t) = \phi(|\mathbf{k}|^2/2, t)$ (isotropic solution) then

$$|\mathbf{k}_+|^2 = z^2\,|\mathbf{k}|^2\,\mu^2, \qquad |\mathbf{k}-\mathbf{k}_+|^2 = |\mathbf{k}|^2\,[1-z(2-z)\,\mu^2], \qquad \mu = \mathbf{k}\cdot\mathbf{n}/|\mathbf{k}|.$$

and we obtain for $n \geqslant 2$

$$\mathfrak{I}(\phi, \phi) = \int_0^1 ds\, G_n(s)\{\phi(z^2 sx)\,\phi[(1-\beta s)\,x] - \phi(0)\,\phi(x)\}, \tag{2.20}$$

where

$$x = |\mathbf{k}|^2/2, \quad \beta = z(2-z), \quad G_n(s) = \tfrac{1}{2}\,\Omega_{n-2}(1-s)^{\frac{n-3}{2}}, \quad n = 2, 3,... \tag{2.21}$$

and Ω_{n-1} denotes the "area" of the unit sphere in n-dimensions, S^{n-1} ($\Omega_1 = 2\pi,...$).

Hence, in the case of isotropic solutions, Eq. (2.18) becomes:

$$\frac{\partial\phi}{\partial t} = \int_0^1 ds\, G_n(s)\{\phi(z^2 sx)\,\phi[(1-\beta s)\,x] - \phi(0)\,\phi(x)\},$$
$$x = |\mathbf{k}|^2/2, \quad n = 2, 3,... \tag{2.22}$$

This equation coincides with the usual (elastic) Fourier-transformed Boltzmann equation, if $z = \beta = 1$. The case $n = 1$

$$\frac{\partial\phi}{\partial t}(|k|, t) = \phi(z|k|)\,\phi[(1-z)\,|k|] - \phi(0)\,\phi(|k|) \tag{2.23}$$

is formally quite similar to the purely elastic case with $z = 1$ and $G_1(s) = \delta(s - s_0)$, with an obvious change of notation. There is, however, an essential difference between the two cases, since Eq. (2.22) is written for $\phi(|\mathbf{k}|^2/2, t)$ whereas Eq. (2.23) is valid for $\phi(|k|, t)$. This difference plays an important role when we invert the Fourier transform. In particular the well-known BKW solution[18, 19] of Eq. (2.23)

$$\phi(|k|, t) = \psi_{\mathrm{BKW}}(|k|\,e^{-\mu t}), \quad \psi_{\mathrm{BKW}}(x) = e^{-x}(1+x), \quad \mu = z(1-z) \tag{2.24}$$

leads to the distribution function

$$f(|v|, t) = e^{\mu t} F(|v|\,e^{\mu t}), \qquad F(|v|) = \frac{1}{2\pi}\int_{-\infty}^{\infty} dk\, e^{ikv - |k|}(1 + |k|) = \frac{2}{\pi(1+|v|^2)^2}. \tag{2.25}$$

with a power-like tail for $|v| \to \infty$. This solution was originally derived in ref. 5 without noting the relation to to the BKW mode. When combined with some results of numerical experiments,[5] it was probably the starting point for the "Ernst–Brito conjecture" (see refs. 3 and 4 for more details).

Remark 2.1. The BKW mode was first published in ref. 18 as the general class of distribution functions having the form

$$f(|\mathbf{v}|, t) = (a(t) + b(t)\,|\mathbf{v}|^2)\, e^{-c(t)\,|\mathbf{v}|^2},$$

This class is obviously invariant under the Fourier transform

$$\phi(x, t) = (f, e^{-i\mathbf{k}\cdot\mathbf{v}}) = (A(t) + B(t)\,x)\, e^{-C(t)\,x}, \qquad x = |\mathbf{k}|^2.$$

One can easily check that the whole class of such solutions of Eq. (2.22) (with $z = \beta = 1$) normalized by the condition $\phi(0, t) = 1$ reads

$$\phi(x, t) = e^{-\alpha x}\psi_{\mathrm{BKW}}(\beta x e^{-\mu t}), \qquad \int_0^1 ds\, s(1-s)\, G(s)$$

where α and β are free parameters. This form of the solution was used in most publications based on the Fourier transform, published after 1975.[16]

We consider below both elastic and inelastic Maxwell models. The elastic case (with arbitrary kernel $G(s) \geqslant 0$ and $z = \beta = 1$) is discussed in Section 3.

3. ELASTIC BOLTZMANN EQUATION AND ITS SELF-SIMILAR SOLUTIONS WITH FINITE ENERGY

The Cauchy problem for the elastic Boltzmann equation (pseudo-Maxwell molecules) reads:

$$\frac{\partial \phi}{\partial t} = \int_0^1 ds\, G(s)\{\phi(sx)\,\phi[(1-s)\,x] - \phi(0)\,\phi(x)\}, \tag{3.1}$$

$$\phi_{|t=0|} = \phi_0(x), \qquad \phi_0(0) = \phi(0, t) = 1, \qquad x \geqslant 0, \quad t > 0,$$

where the initial characteristic function is arbitrary:

$$\phi_0\left(\frac{|\mathbf{k}|^2}{2}\right) = \int_{\mathbb{R}^n} f_0(|\mathbf{v}|)\, e^{-i\mathbf{k}\cdot\mathbf{v}}\, d\mathbf{v}, \qquad \mathbf{k} \in \mathbb{R}^n, \quad n = 2, 3, \ldots \tag{3.2}$$

Here $f_0(|\mathbf{v}|)$ is a (generalized) density of a probability measure in $\mathbb{R}^n$. It is well-known (see ref. 2 for details) that in this case there exists a unique characteristic function $\phi(x, t)$ satisfying Eq. (3.1). We recall that the pointwise convergence $\phi_n \to \phi_\infty$ of characteristic functions is equivalent to the convergence $f_n \to f_\infty$ of the corresponding distribution functions in the sense of probability measures.[15]

We distinguish two essentially different classes of initial conditions:

$$\text{(A)} \quad \frac{1}{n}(f_0, |\mathbf{v}|^2) = -\phi_0'(0) = 1 \qquad \text{(finite energy)} \tag{3.3}$$

and

$$\text{(B)} \quad (f_0, |\mathbf{v}|^2) = \infty \qquad \text{(infinite energy).} \tag{3.4}$$

In case (A) the solution $f(|\mathbf{v}|, t)$ of EBE tends, as $t \to \infty$ to the Maxwell distribution. This means that

$$\phi(x, t) \to e^{-x}, \qquad t \to \infty, \tag{3.5}$$

where $\phi(x, t)$ is the solution of Eq. (3.1).

A long time asymptotics of $\phi(x, t)$ for case (B) is much less trivial. It was proved recently[2] that for certain classes of initial conditions we obtain a typical self-similar asymptotics

$$\phi(xe^{-at}, t) \to \psi_a(x), \qquad t \to \infty, \tag{3.6}$$

where the value of the parameter $a > 0$ depends on the initial condition. We note that the characteristic function $\phi_a(x, t) = \psi_a(xe^{at})$ is itself a self-similar solution of Eq. (3.1). The whole class of such self-similar solutions was constructed in ref. 2; moreover, some of them (for $G = 1$ in (3.1)) were found in an explicit form.[1, 2]

The results, however, refer to case (B) (infinite energy) which is less important for applications than the class (A) (finite energy). We shall see below that some of the results of ref. 2 concerning the self-similar asymptotics can be generalized to the case (A) for both elastic and inelastic Maxwell models.

Let us consider the Cauchy problem (3.1) with the condition (3.3) (case (A)). Then we let

$$\phi(x, t) = e^{-x}\tilde{\phi}(x, t), \qquad \phi_0(x) = e^{-x}\tilde{\phi}_0(x), \tag{3.7}$$

and assume that

$$\tilde{\phi}_0(x) = 1 + O(x^p), \qquad x \to 0, \quad p > 1. \tag{3.8}$$

The equation for $\tilde{\phi}(x, t)$ and the condition $\tilde{\phi}(0, t) = 1$ remain unchanged. Therefore we omit tildes below and consider problem (3.1) again, with the additional assumption

$$\phi_0(x) = 1 + ax^p + \cdots, \qquad x \to 0, \quad p > 1. \tag{3.9}$$

Then a solution for $x \to 0$ reads

$$\phi(x) = 1 + a(xe^{-\mu_p t})^p + \cdots, \tag{3.10}$$

where

$$\mu_p = \frac{\lambda(p)}{p}, \qquad \lambda(p) = \int_0^1 ds\, G(s)[1 - (s)^p - (1-s)^p], \qquad p > 1. \tag{3.11}$$

A self-similar asymptotics would mean that

$$\phi(xe^{\mu_p t}, t) \to \psi_p(x), \qquad t \to \infty, \tag{3.12}$$

where $\psi(xe^{-\mu_p t})$ is a self-similar solution of Eq. (3.1). Hence, we start a more detailed study of the whole class of self-similar solutions satisfying for a fixed p the equation

$$\begin{gathered} \mu_p x\psi'(x) + \Phi(x) - \psi(x) = 0, \\ \Phi(x) = \int_0^1 ds\, G(s)\, \phi(sx)\, \phi[(1-s)\, x], \end{gathered} \tag{3.13}$$

where it is assumed without loss of generality that

$$\int_0^1 ds\, G(s) = 1. \tag{3.14}$$

The following properties of $\lambda(p)$ are well known:[14]

$$0 < \lambda(p) < 1,\ \lambda(1) = 0, \qquad \lambda'(p) > 0, \qquad \lambda''(p) < 0, \qquad p \geqslant 1. \tag{3.15}$$

Therefore

$$[p\lambda'(p) - \lambda(p)]' = p\lambda''(p) < 0, \tag{3.16}$$

and this leads to the following properties of the function $\mu_p = \mu(p)$ defined by (3.11) (note that $p^2\mu'(p) = p\lambda'(p) - \lambda(p)$):

(a) $\mu(p) > 0$ if $p > 1$, $\mu(1) = 0$, $\lim_{p \to \infty} \mu(p) = 0$,

(b) $\mu(p)$ has a unique maximum at, say, $p = p_*$; moreover, $2 < p_* < 3$ since $\mu(2) = \mu(3)$ and $p_* \lambda'(p_*) = \lambda(p_*)$.

A typical behavior of $\mu(p)$ can be understood by considering the simplest case $G = 1$, for which:

$$\mu(p) = \frac{p-1}{p(p+1)}, \qquad p_* = 1 + \sqrt{2}. \tag{3.17}$$

Hence the equation

$$\mu(p) = b = \text{const.}, \qquad p > 1, \tag{3.18}$$

has:

(1) no roots if $b > \mu(p_*)$;

(2) a unique double root $p = p_*$ if $b = \mu(p_*)$;

(3) two different roots $p_1 < p_* < p_2$ if $0 < b < \mu(p_*)$.

We consider the most important case (3) and first assume that p_1 and p_2 are rationally independent. Then the solution of Eq. (3.13) can be constructed as a double power series

$$\psi(x) = \sum_{n,\, m = 0}^{\infty} b(n, m)\, x^{np_1 + mp_2}, \qquad b(0, 0) = 1, \quad 1 < p_1 < p_* < p_2. \tag{3.19}$$

Here the two parameters $b(1, 0)$ and $b(0, 1)$ are arbitrary whereas all the other coefficients can be found by recurrence formulas after substituting the series into Eq. (3.13). If p_1 and p_2 are rationally dependent, say $kp_1 = lp_2$ for some integers $k > l$ having no common prime factors, then Eq. (3.19), should be replaced by

$$\psi(x) = 1 + \sum_{n = l}^{\infty} b_n x^{n\alpha}, \qquad \alpha = \frac{p_1}{l} = \frac{p_2}{k},$$

where we still have two free parameters b_l and b_k. This is, by the way, the case of the BKW solution (2.24), where $p_1 = 2$, $p_2 = 3$, $\alpha = 1$.[14, 17]

We consider the problem (3.1) again, with $\phi_0(x)$ satisfying (3.7) for a given $p > 1$. If the conjecture (3.12) is true and $1 < p < p_*$ then we must choose $p_1 = p$, $b(1, 0) = a$ because of the asymptotic equality (3.10) (as $x \to 0$). It remains, however, unclear how to choose the second parameter $b(0, 1)$ in the series (3.19) for a given initial data $\phi_0(x)$. If $p > p_*$ then the uncertainty disappears since the only option is to choose $p_2 = p$,

$b(0, 1) = a$, $b(1, 0) = 0$. Then we obtain a simplified, one-parameter self-similar solution (it is easy to show that $b(n, m) = 0$ for any $n \geqslant 1$ if $b(1, 0) = 0$):

$$\psi(x) = \sum_{m=0}^{\infty} b_m x^{mp}, \qquad b_0 = 1, \qquad b_1 = a, \qquad p > p_*. \tag{3.20}$$

Such solutions for $p = 3, 4, \ldots$ were studied long ago[17] and soon it became clear that they cannot play any role in the long time asymptotics. Hence the conjecture (3.12) is not true in the general case. We shall see, however, that there is a case (especially important for the inelastic models) when the conjecture is true.

Remark 3.1. One can easily guess from a comparison of Eqs. (2.22) and (3.1) that practically the same considerations can be made for the case of IBE. Moreover, no changes are needed if one considers the 1d case in the variable $x = |k|$ instead of $x = |\mathbf{k}|^2/2$ (Eq. (2.23) is the particular case of Eq. (3.1) with $G(s) = \delta(s - z)$).

It is easy to show that the solution $\phi(x, t)$ of the problem (3.1) has the following property: If $0 \leqslant \phi_0 \leqslant 1$ then $0 \leqslant \phi(x, t) \leqslant 1$ for all $t > 0$. Is, perhaps, the conjecture true at least for these solutions? In order to clarify this question we consider the initial data satisfying the following two conditions:

$$0 \leqslant \phi_0(x) \leqslant 1; \qquad \phi_0(x) = 1 - \tfrac{1}{2} x^p + \cdots, \qquad x \to 0, \quad p > 1. \tag{3.21}$$

where the second condition is obtained from the general case (3.9) with $a < 0$ by the scaling transformation $x \to |a|^{-1/p} x$ which does not change Eqs. (3.1) and (3.13).

It is convenient to change the x-variable by the transformation:

$$\phi_0(x) = \hat{\phi}_0(x^\theta), \qquad \phi(x, t) = \hat{\phi}(x^\theta, t), \qquad \theta = p/2 \tag{3.22}$$

and then omit hats in the final equation. Thus we transform the problem (3.1) to

$$\frac{\partial \phi}{\partial t} = I_\theta(\phi) - \phi(x), \qquad I_\theta(\phi) = \int_0^1 ds\, G(s)\, \phi(s^\theta x)\, \phi[(1-s)^\theta x],$$
$$\phi_{|t=0|} = \phi_0(x), \quad 0 \leqslant \phi_0(x) \leqslant 1, \quad \phi_0(x) = 1 - \tfrac{1}{2} x^2 + \cdots, \quad x \to 0, \quad \theta > \tfrac{1}{2}. \tag{3.23}$$

The corresponding equation for self-similar solutions $\phi(x, t) = \psi(xe^{-t/\gamma})$ now reads

$$\gamma^{-1} x\psi(x) + I_\theta(\psi) - \psi = 0, \qquad \gamma^{-1} = \tfrac{1}{2}\,\lambda(2\theta), \qquad \theta > \tfrac{1}{2}. \tag{3.24}$$

In the next section we shall show that Eqs. (3.23) and (3.24) with $1 < \theta < 2$ have common properties with the Fourier transformed isotropic IBE (2.22) and can, therefore, be studied in a similar way.

4. UNIFIED APPROACH TO EBE AND IBE

We transform Eq. (2.22) with an arbitrary kernel $G(s) \geqslant 0$ by introducing a new function $\hat{\phi}$

$$\hat{\phi}(x, t) = \phi(\sqrt{x}, t). \tag{4.1}$$

Then we make the necessary scaling transformations $\hat{x} = x/x_0$, $\hat{t} = t/x_0$ and omit hats in the final equations. This leads to the following Cauchy problem

$$\begin{gathered}\frac{\partial \phi}{\partial t} = \mathscr{I}(\phi) - \phi, \qquad \mathscr{I}(\phi) = \int_0^1 ds\, G(s)\, \phi(\sqrt{s}\, zx)\, \phi[\sqrt{1-\beta s}\, x], \\ \phi_{|t=0|} = \phi_0(x), \qquad |\phi_0(x)| \leqslant 1, \qquad \phi_0(x) = 1 - \frac{1}{2}\, x^2 + \cdots, \qquad x \to 0.\end{gathered} \tag{4.2}$$

It was assumed above that the condition (3.14) is satisfied. The restriction $|\phi_0(x, t)| \leqslant 1$ is automatically fulfilled for any characteristic function. Usually we replace it by a weaker condition

$$0 \leqslant \phi_0(x) \leqslant 1 \tag{4.3}$$

which is also preserved in time. We note that a typical initial condition is a Maxwellian

$$\phi_0(x) = \exp(-x^2/2). \tag{4.4}$$

If the initial distribution function $f_0(|\mathbf{v}|)$ decays for $|\mathbf{v}| \to \infty$ faster than $\exp(-\alpha\, |\mathbf{v}|)$, then $\phi_0(x)$ is represented by the following series

$$\phi_0(x) = 1 - \tfrac{1}{2}\, x^2 + \sum_{n=2}^{\infty} (-1)^n\, a_n x^{2n}, \qquad a_n > 0, \tag{4.5}$$

with a non-zero radius of convergence.

By using Eqs. (2.13), (2.21), and (4.1) one can easily verify the identity

$$(f_0, |\mathbf{v}|^2) = -(d/2)\,\phi_0''(0).$$

This explains the following

Remark 4.1. The (formally) general case:

$$\phi_0(x) = 1 - ax^{2\theta} + \cdots, \qquad x \to 0, \quad \theta \neq 1, \tag{4.6}$$

is not very relevant for IBE since: (a) $\theta > 1$ means that $f_0(|\mathbf{v}|)$ is partly negative (zero energy), and (b) $\theta < 1$ means that $f_0(|\mathbf{v}|)$ has infinite energy and in such a case (much less interesting for applications) we have an extra difficulty in making the inelastic Maxwell model consistent with the initial hard sphere model (see Eqs. (2.10) and (2.17)). This is the reason why we consider for IBE just the value $\theta = 1$ at variance with case $e = z = 1$ (Section 3).

The Cauchy problem (4.2) leads (as in Section 3) to the following equation for self-similar solutions $\phi(x, t) = \psi(xe^{-t/\delta})$

$$\begin{gathered} \delta^{-1} x\psi'(x) + \mathscr{I}(\psi) - \psi = 0, \\ \delta^{-1} = \tfrac{1}{2} \int_0^1 ds\, G(s)[1 - sz - (1 - \beta s)] = \tfrac{1}{2}\, z(1-z) \int_0^1 ds\, G(s)\, s \end{gathered} \tag{4.7}$$

The similarity of Eqs. (3.23), (3.24) and (4.2), (4.7) is obvious. Therefore we consider below a more general equation

$$\frac{\partial \phi}{\partial t} = \mathscr{K}(\phi) - \phi, \qquad \mathscr{K}(\phi) = \int_0^1 ds\, G(s)\, \phi[a(s)\, x]\, \phi[b(s)\, x], \tag{4.8}$$

with arbitrary functions $0 \leqslant a(s), \beta(s) \leqslant 1$ and $G(s)$ satisfying the condition (3.14). We also assume that

$$\begin{gathered} a^2(s) + b^2(s) \leqslant 1, \qquad 0 \leqslant s \leqslant 1, \\ \lambda = \int_0^1 ds\, G(s)[1 - a^2(s) - b^2(s)] > 0, \end{gathered} \tag{4.9}$$

and note that the case

$$a(s) = s^\theta \qquad b(s) = (1-s)^\theta, \qquad \theta > \tfrac{1}{2}, \tag{4.10}$$

corresponds to EBE (3.23), whereas the case

$$a(s) = z\sqrt{s}, \qquad b(s) = \sqrt{1-\beta s}, \qquad \tfrac{1}{2} \leqslant z \leqslant 1, \quad \beta = z(2-z), \tag{4.11}$$

corresponds to IBE (4.2). The initial condition satisfies our usual restrictions:

$$\phi_{|t=0|} = \phi_0(x), \quad 0 \leqslant \phi_0(x) \leqslant 1, \quad \phi_0(x) = 1 - \tfrac{1}{2}x^2 + \cdots, \qquad x \to 0. \tag{4.12}$$

The equation for self-similar solutions $\phi(x, t) = \psi(xe^{-t/r})$ now reads

$$r^{-1}x\psi'(x) + \mathscr{K}(\psi) - \psi = 0, \qquad r^{-1} = \frac{\lambda}{2}, \tag{4.13}$$

where

$$\psi(x) = 1 - \tfrac{1}{2}x^2 + \cdots, \qquad x \to 0. \tag{4.14}$$

The assumption of self-similar asymptotics, i.e.,

$$\phi(xe^{t/r}, t) \to \psi(x), \qquad t \to \infty, \tag{4.15}$$

may be true only if $0 \leqslant \psi(x) \leqslant 1$. Hence, we have to study only bounded positive solutions of Eq. (4.13) satisfying (4.14). The simplest explicit example of such solution corresponds to the BKW-solution of the Boltzmann equation: if $\theta = 1$ in Eq. (3.24), then $\psi_{\mathrm{BKW}}(x)$ (2.24) is a solution satisfying all the above conditions. How can we generalize this simple solution to the case of the operator $K[\phi]$ with arbitrary functions $a(s)$ and $b(s)$? This question is considered in the next section.

5. AN INTEGRAL EQUATION FOR THE SELF-SIMILAR SOLUTIONS

Equation (4.13) can be rewritten as

$$\begin{gathered} \frac{d}{dx}[x^{-r}\psi(x)] = -\frac{r}{x^{r+1}}F(x), \\ F(x) = \mathscr{K}(\psi) = \int_0^1 ds\, G(s)\, \psi[a(s)\, x]\, \psi[b(s)\, x]. \end{gathered} \tag{5.1}$$

Assuming that $\psi(x)$ is bounded we therefore obtain the following integral equation:

$$\psi(x) = R[\psi] = rx^r \int_x^\infty \frac{dy}{y^{r+1}} F(y) = r \int_1^\infty \frac{d\tau}{\tau^{r+1}} F(\tau x), \tag{5.2}$$

$$r^{-1} = \frac{1}{2} \int_0^1 ds\, G(s)[1 - a^2(s) - b^2(s)]. \tag{5.3}$$

This equation has obviously two trivial solutions $\psi = 0$ and $\psi = 1$. Our aim is to construct a nontrivial solution satisfying (4.14). A natural way is to use an iteration process:

$$\psi_{n+1}(x) = R[\psi_n], \qquad n = 0, 1, \ldots \tag{5.4}$$

We note that: (1) $0 \leqslant R[\psi] \leqslant 1$ if $0 \leqslant \psi \leqslant 1$, (2) $R[\psi] \leqslant R[\phi]$ if $0 \leqslant \psi \leqslant \phi$. Hence the iteration converges point-wise if we choose the initial approximation $0 \leqslant \psi_0 \leqslant 1$ in such a way that

$$\psi_0(x) \leqslant R[\psi_0] \qquad \text{or} \qquad \psi_0(x) \geqslant R[\psi_0]. \tag{5.5}$$

An obvious choice is $\psi_0(x) = \psi_{\mathrm{BKW}}$, defined by (2.24). The following observation shows that this choice does lead to convergence.

Lemma 5.1. If $\psi(x) = e^{-x}(1+x)$ then

(1) $\psi \leqslant R[\psi]$ if $a(s) + b(s) \leqslant 1$, a.e. for $0 \leqslant s \leqslant 1$;

(2) $\psi \geqslant R[\psi]$ if $a(s) + b(s) \geqslant 1$, a.e. for $0 \leqslant s \leqslant 1$;

(3) $\psi = R[\psi]$ if $a(s) + b(s) = 1$, a.e. for $0 \leqslant s \leqslant 1$.

Proof. If we denote

$$\Delta(x) = \frac{\lambda}{2} x\psi'(x) + \mathscr{K}(\psi) - \psi, \qquad \lambda = \int_0^1 ds\, G(s)[1 - a^2(s) - b^2(s)],$$

then

$$R[\psi] - \psi = r \int_1^\infty \frac{d\tau}{\tau^{r+1}} \Delta(\tau x). \tag{5.6}$$

Therefore it is sufficient to show that $\Delta \geqslant 0$ in case (1), $\Delta \leqslant 0$ in case (2), and $\Delta = 0$ in case (3). We can represent $\Delta(x)$ as

$$\Delta(x) = \int_0^1 ds\, G(s)\, g[x; a(s), b(s)], \tag{5.7}$$

where

$$g(x;\, a,\, b) = e^{-(a+b)\,x}(1+ax)(1+bx) - e^{-x}\left(1+x+\frac{1-a^2-b^2}{2}\,x^2\right)$$

First we assume that $a+b \leqslant 1$. Then we perform an elementary estimate omitting obviously positive terms:

$$\begin{aligned} g(x;\, a,\, b) &= e^{-x}\{[e^{[1-(a+b)]\,x}(1+ax)(1+bx) - (1+x+cx^2)\} \\ &\geqslant e^{-x}\left\{[1+(1-a-b)]\,x + \frac{(1-a-b)^2}{2}\,x^2(1+ax)(1+bx)\right. \\ &\quad \left. -(1+x+cx^2)\right\} \\ &\geqslant e^{-x}x^2\left[ab+(1-a-b)(a+b)+\frac{(1-a-b)^2}{2}-c\right]=0, \\ c &= \frac{1-a^2-b^2}{2} \end{aligned}$$

and hence (1) is proved. If $a+b \geqslant 1$, then

$$\begin{aligned} -g(x;\, a,\, b) &= e^{-(a+b)\,x}[e^{(a+b-1)\,x}(1+ax)(1+bx) - (1+x+cx^2)\} \\ &\geqslant e^{-(a+b)\,x}x^2\left[c+\frac{(a+b-1)^2}{2}+(a+b-1)-ab\right]=0, \end{aligned}$$

and (2) is proved. (3) is obvious. ▮

Hence, the sequence $\{\psi_n(x)\}$, defined by Eq. (5.4) with $\psi_0 = e^{-x}(1+x)$, is: (1) monotone increasing if $a(s)+b(s) \leqslant 1$, and (2) monotone decreasing if $a(s)+b(s) \geqslant 1$. The convergence is guaranteed since $0 \leqslant \psi_n \leqslant 1$ for all $n = 0, 1, \ldots$. The only remaining problem is to exclude the trivial limits

$$\text{(a)}\quad \lim_{n\to\infty} \psi_n(x) = 1, \qquad \text{(b)}\quad \lim_{n\to\infty} \psi_n(x) = 0. \tag{5.8}$$

Then it is clear that the function

$$\psi(x) = \lim_{n\to\infty} \psi_n(x) \tag{5.9}$$

is a nontrivial solution of the integral equation.

First we consider case (2) of Lemma 5.1 assuming that $a+b \geqslant 1$. Then the iteration process (5.4) leads to a monotone decreasing sequence

$$\psi_0 = e^{-x}(1+x), \qquad \psi_1 = R[\psi_0] \leqslant \psi_0, \qquad \psi_2 = R[\psi_1] \leqslant \psi_1, \dots . \tag{5.10}$$

If we find a non-zero function $w(x) \geqslant 0$ such that

$$\text{(a)} \quad w(x) \leqslant e^{-x}(1+x), \qquad \text{(b)} \quad R[w] \geqslant w, \tag{5.11}$$

then we get the estimate

$$\psi_n(x) \geqslant w(x), \qquad n = 0, 1, \dots, \dots;$$

thus excluding the trivial limit $\psi(x) = 0$. An example of such function is given by

Lemma 5.2. The function $w(x) = \psi_M = \exp(-x^2/2)$ satisfies both conditions (a) and (b) in (5.11), provided

$$a^2(s) + b^2(s) \leqslant 1, \qquad \text{a.e. for} \quad 0 \leqslant s \leqslant 1.$$

Proof. Condition (a) is satisfied since

$$x - \log(1+x) \leqslant x^2/2, \qquad x > 0.$$

Condition (b) can be reduced to the inequality $\Delta_{\psi_M} \geqslant 0$ in the notation of Eq. (5.6) with ψ_M replacing ψ. In a way similar to that used for Eq. (5.7), we obtain

$$\Delta_{\psi_M}(x) = \int_0^1 ds\, G(s)\, h[x; a(s), b(s)],$$

where

$$\begin{aligned} h(x; a, b) &= e^{-(a^2+b^2)\,x^2/2} - e^{-x^2/2}\left(1 + \frac{1-a^2-b^2}{2}\,x^2\right) \\ &= e^{-x^2/2}\left[e^{(1-a^2-b^2)\,x^2/2} - \left(1 + \frac{1-a^2-b^2}{2}\,x^2\right)\right] \geqslant 0 \end{aligned}$$

Hence $\Delta_{\psi_M} \geqslant 0$ and therefore $R[\psi_M] \geqslant \psi_M$ (see Eq. (5.6)). ∎

Lemmas 5.1 and 5.2 in combination with the monotone iteration scheme (5.10) prove the following

Theorem 5.3. The integral equation (5.2) has a non-trivial solution $\psi(x)$ such that

$$e^{-x^2/2} \leqslant \psi(x) \leqslant e^{-x}(1+x), \qquad x \geqslant 0, \tag{5.12}$$

provided

$$a(s)+b(s) \geqslant 1, \quad \text{and} \quad a^2(s)+b^2(s) \leqslant 1, \quad \text{a.e. for} \quad 0 \leqslant s \leqslant 1. \tag{5.13}$$

The above theorem applies to the case of EBE (3.23) with $\frac{1}{2} < \theta \leqslant 1$ and to the case of IBE (4.2) when $a(s)$ and $b(s)$ are given by Eqs. (4.11). To prove the latter case we need to verify that

$$q(z, s) = z\sqrt{s} + \sqrt{1 - z(2-z)\, s} \geqslant 1, \qquad 0 \leqslant s \leqslant 1, \quad \tfrac{1}{2} \leqslant z \leqslant 1.$$

The function $q(z, s)$ has the following properties:

$$(1) \quad q(z, 0) = q(z, 1) = 1; \qquad (2) \quad q_{ss}(z, s) < 0, \qquad 0 < s < 1, \quad 1/2 \leqslant z \leqslant 1.$$

Hence, for any fixed $1/2 \leqslant z \leqslant 1$, the function $q(z, s)$ as a function of s cannot have a local minimum on $(0, 1)$ and therefore $q(z, s) \geqslant 1$.

Remark 5.4. The case when $a(s)+b(s) \leqslant 1$ is technically more difficult and less important for applications. There is almost no hope, in this case, to prove a "simple" result similar to Theorem 5.3. In the case of EBE (3.23) with $\theta > 1$ in Eqs. (4.10), such a result would be valid also for $\theta > \theta_* = p_*/2$, when the corresponding power series (3.19) actually does not have any free parameters (after the scaling transformation $\hat{x} = ax$). This would mean that the function $\psi(x)$ represented by the series (3.19) is automatically bounded for all $x > 0$, which seems very doubtful. Therefore we do not consider the case $a(s)+b(s) \leqslant 1$ in the sequel and hope to clarify it completely in a future paper.

In the next section we study the role of self-similar solutions for large time asymptotics.

6. SELF-SIMILAR ASYMPTOTICS

Let us consider Eq. (4.8) with initial condition (4.12). We assume that $G(s) \geqslant 0$, $0 \leqslant a(s), b(s) \leqslant 1$ satisfy conditions (3.14), (4.9) and

$$a(s)+b(s) \geqslant 1, \qquad s > 0. \tag{6.1}$$

In accordance with Theorem 5.3 there exists a self-similar solution $\phi_S(x, t) = \psi(xe^{-\lambda t/2})$, satisfying

$$0 \leqslant \psi(x) \leqslant 1; \qquad \psi(x) = 1 - \tfrac{1}{2}x^2 + \cdots, \qquad x \to 0. \tag{6.2}$$

We denote

$$\phi(x, t) = \psi(xe^{-\lambda t/2}) - u(x, t), \qquad \phi_0(x) = \psi(x) - u_0(x), \tag{6.3}$$

where $\phi(x, t)$ is the solution of the Cauchy problem (4.8), (4.12). Such a solution can be constructed, for example, in the form of a standard Wild's sum.[2]

The initial value problem for $u(x, t)$ reads as follows

$$\frac{\partial u}{\partial t} = P(U, u) + P(u, U) - u, \tag{6.4}$$

where

$$\begin{aligned} P(u_1, u_2) &= \int_0^1 ds\, G(s)\, u_1[a(s)\, x]\, u_2[b(s)\, x], \\ U(x, t) &= \frac{1}{2}[\phi(x, t) + \psi(xe^{-\lambda t/2})], \end{aligned} \tag{6.5}$$

and

$$u_{|t=0|} = u_0(x) = \psi(x) - \phi_0(x). \tag{6.6}$$

We consider the *linear* equation (6.4) for $u(x, t)$ assuming that $U(x, t)$ is given. If $u_{1,2}(x, t)$ are two solutions of Eq. (6.4) and $u_1(x, 0) \leqslant u_2(x, 0)$, then obviously $u_1(x, t) \leqslant u_2(x, t)$ for all $t \geqslant 0$, since $U(x, t) \geqslant 0$. Hence, $|u(x, t)| \leqslant \tilde{u}(x, t)$, where $\tilde{u}(x, t)$ is the solution of (6.4) such that $\tilde{u}(x, 0) = |u_0(x)|$. Noting that $0 \leqslant U(x, t) \leqslant 1$, we obtain the following estimate

$$\begin{aligned} &|u(x, t)| \leqslant e^{tL}\, |u_0(x)|, \qquad t \geqslant 0, \\ &Lu = \int_0^1 ds\, G(s)\{u[a(s)\, x] + u[b(s)\, x] - u(x)\}, \end{aligned} \tag{6.7}$$

It is clear that

$$e^{tL}u(x) \geqslant 0 \qquad \text{if} \quad u \geqslant 0$$

(note that $L=L_{+}-I$, where $L_{+}u\geqslant 0$ for any $u\geqslant 0$). Moreover, for any $\alpha>0$

$$LS_\alpha = S_\alpha L, \qquad S_\alpha u(x)=u(\alpha x).$$

Therefore

$$|u(xe^{\mu t},t)|\leqslant e^{tL}\,|u_0(xe^{\mu t})|, \qquad \mu=\frac{\lambda}{2}=\frac{1}{2}\int_0^1 ds\,G(s)[1-a^2(s)-b^2(s)].$$

We note that

$$Lx^{2p}=-\lambda(p)\,x^{2p},\ \lambda(p)=\int_0^1 ds\,G(s)\{[1-[a(s)]^{2p}-[b(s)]^{2p}\},$$

Let us assume now that

$$|u_0(x)|\leqslant \beta x^{2(1+\epsilon)}, \qquad \beta>0, \quad \epsilon>0. \tag{6.8}$$

Then

$$0\leqslant |u(xe^{\mu t},t)|\leqslant \beta x^{2(1+\epsilon)}e^{-\gamma(\epsilon)\,t}, \tag{6.9}$$

$$\gamma(\epsilon)=\lambda(1+\epsilon)-(1+\epsilon)\,\lambda(1)=\frac{1}{2}\int_0^1 ds\,G(s)\,B[2(1+\epsilon);a(s),b(s)], \tag{6.10}$$

$$B(p;a,b)=2(1-a^p-b^p)-p(1-a^2-b^2).$$

Our goal is to prove that $\gamma(\epsilon)>0$ for sufficiently small $\epsilon>0$ provided the conditions (3.14), (4.10), and (6.1) are fulfilled. We note that

$$\frac{\partial B}{\partial b}=2pb(1-b^{p-2})\geqslant 0, \qquad p\geqslant 2.$$

Since $b\geqslant 1-a$ thanks to Eq. (6.1), we have the estimate

$$B(p;a,b)\geqslant g(p;a), \qquad g(p;a)=B(p;a,1-a).$$

The function

$$g(p;a)=2[1-a^p-(1-a)^p]-p[1-a^2-(1-a)^2]$$

is, for any fixed $a \in (0, 1)$, a concave function of p since

$$\frac{\partial^2 g}{\partial p^2} = -2[(\log a)^2 a^p + (\log(1-a))^2 (1-a)^p] \leqslant 0, \qquad 0 < a \leqslant 1 \tag{6.11}$$

On the other hand, $g(2; a) = g(3; a) = 0$. Therefore

$$g(p; 0) = g(p; 1) = 0, \qquad g(p; a) \geqslant 0 \quad \text{if} \quad 0 < a < 1, 2 \leqslant p \leqslant 3.$$

If $2 < p < 3$ then the function $g(p; a)$ has no zero except at $a = 0$ and $a = 1$ (otherwise, Eq. (6.11) would be contradicted). On the other hand, the condition (4.9)

$$0 < \lambda \leqslant \int_0^1 ds\, G(s)\{1 - a^2(s) - [1 - a(s)]^2\}$$

shows that $G(s)$ cannot be concentrated at points where $a(s) = 0$ or $a(s) = 1$. This finally leads to the inequality

$$\gamma(\epsilon) > 0, \qquad 0 < \epsilon < 1/2$$

in the notation of Eq. (6.10). Hence we have proved the following

Lemma 6.1. For any $G(s) \geqslant 0$, $a(s) \geqslant 0$ and $b(s) \geqslant 0$ $(0 \leqslant s \leqslant 1)$ satisfying conditions (3.14), (4.9), (6.1) the function $\gamma(\epsilon)$ (6.10) is strictly positive for $0 < \epsilon < 1/2$.

Let us now return to inequalities (6.8), (6.9) and note that $0 \leqslant |u_0(x)| \leqslant 1$. Therefore the inequality (6.8) is automatically fulfilled for some $0 < \epsilon_0 < 1/2$ if it it is valid for some other $\epsilon > \epsilon_0$ (with another irrelevant constant parameter $\beta > 0$). Therefore we obtain the following result

Theorem 6.2. Let us consider the Cauchy problem (4.8), (4.12) assuming that the conditions (3.14), (4.9), (6.1) are satisfied. In addition we assume that

$$\phi_0(x) = 1 - \tfrac{1}{2} x^2 + O(x^p), \qquad x \to 0, \quad p > 2.$$

Then

$$\phi(x e^{\lambda t/2}, t) \to \psi(x), \qquad t \to \infty$$

for any $x \geqslant 0$.

Proof. We need only to verify (6.8) for $u_0(x) = \psi(x) - \phi_0(x)$, since the rest of the proof has been given above. The estimate (5.12) shows that $\psi(x) = 1 - x^2/2 + O(x^3)$. Hence

$$|u_0(x)| \leqslant |\psi(x) - \phi_0(x)| = |[\psi(x) - (1 - x^2/2)] - [\phi_0(x) - (1 - x^2/2)]|$$
$$= O(x^3) + O(x^p), \quad x \to 0,\, p > 2.$$

On the other hand, $|u_0(x)| \leqslant 1$. Therefore we can always find a positive constant $\beta > 0$ and a positive $\epsilon < \mathrm{Min}(1/2, p/2 - 1)$ and such that the condition (6.8) is fulfilled. Then according to Lemma 6.1, $\gamma(\epsilon) > 0$ in the inequality (6.9) and this completes the proof. ∎

Corollary 6.3. If $\phi_0(x) = \exp(-x^2/2)$ then $\phi(xe^{\lambda t/2}, t) \to \psi(x)$ as $t \to \infty$.

Corollary 6.4. The self-similar solution constructed in Theorem 5.3 is unique in the class of functions satisfying the conditions

$$0 \leqslant \psi(x) \leqslant 1; \qquad \psi(x) = 1 - \tfrac{1}{2} x^2 + O(x^p), \qquad x \to 0, \quad p > 2.$$

The proofs of these two corollaries follow directly from Theorem 6.2. We conclude this section with the following

Lemma 6.5. Let $\psi(x)$ be the function constructed in Theorem 5.3. Then

$$\psi(x) = 1 - \frac{x^2}{2} + \sum_{k=2}^{[p]} a_k x^{2k} + b x^{2p} + \cdots, \qquad x \to 0, \tag{6.12}$$

where dots denote higher order terms and $[p]$ is the integer part of p; $p \geqslant 3/2$ is uniquely defined as the second root (the largest one) of the equation

$$\lambda(p) = p\lambda(1), \qquad \lambda(p) = \int_0^1 ds\, G(s)\{[1 - [a(s)]^{2p} - [b(s)]^{2p}\}, \tag{6.13}$$

Proof. The function $\psi(x)$ satisfies Eq. (4.13) and inequalities (5.12). Hence,

$$\psi(x) = 1 - \frac{x^2}{2} + \psi_2(x); \qquad |\psi_2(x)| \leqslant cx^3, \quad x \to 0. \tag{6.14}$$

Substituting (4.13) into Eq. (4.13) we obtain the leading asymptotic term of $\psi_2(x)$:

$$\psi_2(x) = a_2 x^{2n_2} + \psi_3(x), \qquad \psi_3(x) = o(x^2 n_2), \qquad x \to 0, \quad n_2 = \min(2, p)$$

in the notation of Eq. (6.13). Similarly we obtain

$$\psi_3(x) = a_3 x^{2n_3} + \psi_4(x), \qquad n_3 = \min(3, p),$$

and so on. This proves the expansion (6.12) provided the root of (6.13) does exist. The inequality $p \geqslant 3/2$ follows from Lemma 6.1 since

$$\gamma(\epsilon) = \lambda(1+\epsilon) - (1+\epsilon)\, \lambda(1) > 0, \qquad 0 < \epsilon < 1/2$$

We note that Eq. (6.13) reads

$$\mu = \frac{\lambda(p)}{p} = \lambda(1),$$

and coincides (except for notation) with Eq. (3.18) discussed in Section 3. Arguing as in Section 3, one can easily verify the following properties of $\mu(p)$:

(a) $\mu(p) > 0$, $p \geqslant 1$; (b) $\mu(p) \to 0$, $p \to \infty$; (c) $\mu(p)$ has a unique maximum at $p = p_* > 1$ on $[1, \infty)$.

The property (c) follows from the inequalities $\lambda'(p) > 0$ and $\lambda''(p) < 0$, $p \geqslant 1$. Hence the root $p \geqslant 3/2$ exists and is unique. ▮

Lemma 6.5 actually describes the most general asymptotic expansion (6.11) of a solution of (4.13) satisfying (4.14). All coefficients a_i, with $2 \leqslant i \leqslant [p]$ (if $[p] \neq p$) or $2 \leqslant i \leqslant p-1$ (if $[p] = p$), are defined uniquely by the construction used in Lemma 6.5. We can also assume, without loss of generality, that $a_p = 0$ if $[p] = p$. The coefficient b in Eq. (6.12) for the concrete function $\psi(x)$, defined in Theorem 5.3, cannot be found in such a way. It remains therefore unknown whether or not $b = 0$. We shall see below that $b \neq 0$ in the case of IBE (at least, for almost all values of the restitution coefficient). This, in turn, leads to a power-like high energy tail for the corresponding self-similar solution, as conjectured in refs. 3 and 4.

7. APPLICATION TO THE INELASTIC BOLTZMANN EQUATION

We consider the initial value problem for the n-dimensional distribution function $f(|\mathbf{v}|, t)$ $(n = 2,...)$

$$\frac{\partial f}{\partial t} = Q_i(f, f), \qquad f_{|t=0} = f_0(|\mathbf{v}|), \qquad \mathbf{v} \in \mathbb{R}^n, \quad t > 0, \tag{7.1}$$

where $Q_i(f, f)$ is defined in such a way that the characteristic function

$$\phi\left(\frac{|\mathbf{k}|^2}{2}, t\right) = \int_{\mathbb{R}^n} f(|\mathbf{v}|, t)\, e^{-i\mathbf{k}\cdot\mathbf{v}}\, d\mathbf{v}, \qquad \mathbf{k} \in \mathbb{R}^n, \tag{7.2}$$

satisfies the equation

$$\frac{\partial\phi}{\partial t} = \int_0^1 ds\, G(s)\{\phi(z^2 sx)\, \phi[(1-\beta s)\, x] - \phi(x)\},$$
$$1/2 < z < 1, \quad \beta = z(2-z), \quad x \geqslant 0. \tag{7.3}$$

It is assumed that

$$\phi_0(0) = \phi(0, t) = 1, \qquad G(s) \geqslant 0, \qquad \int_0^1 ds\, G(s) = 1. \tag{7.4}$$

We define a (generalized) solution of (7.1) as a function $f(|\mathbf{v}|, t)$ such that its Fourier transform (7.2) satisfies (7.3) and the initial condition

$$\phi_{|t=0|} = \phi_0\left(\frac{|\mathbf{k}|^2}{2}\right) = \int_{\mathbb{R}^n} f_0(|\mathbf{v}|)\, e^{-i\mathbf{k}\cdot\mathbf{v}}\, d\mathbf{v}. \tag{7.5}$$

Of course we are mainly interested in non-negative solutions having physical meaning. We assume, without loss of generality, that

$$\int_{\mathbb{R}^n} f_0(|\mathbf{v}|)\, d\mathbf{v} = \frac{1}{n} \int_{\mathbb{R}^n} |\mathbf{v}|^2\, f_0(|\mathbf{v}|)\, d\mathbf{v} = 1. \tag{7.6}$$

Our main result for IBE (7.1) is the following

Theorem 7.1. (1) There exists a unique non-negative function $F(|\mathbf{v}|)$ satisfying the conditions (7.6) (with F replacing f_0) such that

$$f_S(|\mathbf{v}|, t) = e^{n\mu t} F(|\mathbf{v}|\, e^{\mu t}), \qquad \mu = z(1-z) \int_0^1 ds\, G(s)\, s, \tag{7.7}$$

satisfies Eq. (7.1). The function $F(|\mathbf{v}|)$ is bounded and continuous on $\mathbb{R}^n$ and is given by equality

$$F(|\mathbf{v}|) = \int_{\mathbb{R}^n} \psi\left(\frac{|\mathbf{k}|^2}{2}\right) e^{i\mathbf{k}\cdot\mathbf{v}}\, \frac{d\mathbf{v}}{(2\pi)^n}, \tag{7.8}$$

where $\psi(x)$ is defined (after proper change of notations) in Theorem 5.3.

(2) There exists a non-empty class A of initial conditions $f_0(|\mathbf{v}|)$ satisfying conditions (7.6) such that the corresponding solutions $f(|\mathbf{v}|, t)$ have the following property

$$e^{-n\mu t} f(|\mathbf{v}|\, e^{-\mu t}, t) \to F(|\mathbf{v}|), \qquad t \to \infty \tag{7.9}$$

where the convergence is understood in the sense of probability measures.

(3) If $f_0(|\mathbf{v}|) \geqslant 0$ satisfies (7.6) and

$$\int_{\mathbb{R}^n} f_0(|\mathbf{v}|)\, |\mathbf{v}|^{2+\epsilon}\, d\mathbf{v} \leqslant \infty \qquad \int_{\mathbb{R}^n} f_0(|\mathbf{v}|)\, e^{i\mathbf{k}\cdot\mathbf{v}}\, d\mathbf{v} \geqslant 0,$$

for some $\epsilon > 0$ and all $\mathbf{k} \in \mathbb{R}^n$, then $f_0 \in A$. In particular, the Maxwellian distribution

$$f_0(|\mathbf{v}|) = M(|\mathbf{v}|) = (2\pi)^{-n/2}\, e^{-|\mathbf{v}|^2/2} \tag{7.10}$$

belongs to the class A.

Proof. The proof is based on Theorems 5.1 and 6.2 after a suitable change of notation (all considerations of Sections 4–6 relate directly to $\hat{\phi}$; see (4.1)) and now we need to reformulate them for the "true" characteristic function $\phi(x, t)$. In particular, Theorem 5.3 relates to a self-similar solution $\phi_S(x, t) = \psi(xe^{-\mu t})$ of (7.3) and states that

$$e^{-x} \leqslant \psi(x) \leqslant e^{-\sqrt{2x}}(1 + \sqrt{2x}), \qquad x = |\mathbf{k}|^2/2 \geqslant 0, \tag{7.11}$$

We note that e^{-x} is the Fourier transform of a non-negative integrable function. The iteration scheme (5.4) can be rewritten in terms of distribution functions $F_n(|\mathbf{v}|)$ such that for example,

$$F_0 = (2\pi)^{-n/2}\, e^{-|\mathbf{v}|^2/2}, \qquad F_{n+1} = \hat{R}(F_n), \qquad n = 0, 1,...$$

Then all $F_n(|\mathbf{v}|)$ remain non-negative and normalized by conditions (7.6). Hence we obtain a sequence of characteristic functions $\psi_n(x)$ (note that $\psi_{n+1}(x) \geqslant \psi_n(x)$ if $\psi_0(x) = e^{-x}$) which converges point-wise on $[0, \infty]$ to $\psi(x)$. The function $\psi(x)$ is obviously continuous at $x = 0$ (this follows, e.g., from estimates (7.11)) and, therefore $\psi(x)$ is also a characteristic function. Hence, $F(|\mathbf{v}|) \geqslant 0$ in (7.7). Uniqueness of $\psi(x)$ (and, therefore, of $F(|\mathbf{v}|)$) is stated in Corollary 6.4 of Theorem 6.2. The inverse Fourier transform and the local properties of $F(|\mathbf{v}|)$ are justified by the upper estimate in Eq. (7.11). Thus, part (1) of the statement is proved.

In order to prove parts (2) and (3) of the theorem, we recall that point-wise convergence of characteristic functions implies the (weak) convergence of distribution functions (in the usual "kinetic" terminology) in the sense of probability measures.[15] The Maxwellian (7.10) belongs to class A (so that A is non-empty!) because of Corollary 6.3 of Theorem 6.2 (note again that the notations we use now differ from those in Section 6). It follows from known results in the theory of characteristic functions[21] that the conditions given in the third part of the statement are sufficient for Theorem 6.2 to apply. ▮

Theorem 7.1 does not contain any information about the asymptotic behavior of $F(|\mathbf{v}|)$ for large $|\mathbf{v}|$. This behavior ca be obtained on the basis of Lemma 6.5. For the sake of simplicity, we consider here just one question: existence of the moments

$$m_l = (F, |\mathbf{v}|^{2l}). \tag{7.12}$$

of integer order $l = 0, 1, 2, \ldots$. First we note that $m_0 = m_1/d = 1$ by construction. The existence of higher moments is stated in the following

Theorem 7.2. The function $F(|\mathbf{v}|)$ defined in Theorem 7.1 has moments of all orders if and only if the maximal root $p \geqslant 3$ of the equation

$$\int_0^1 ds\, G(s)[1-(z^2 s)^p - (1-\beta s)^p] - 2pz(1-z) \int_0^1 ds\, sG(s) = 0 \tag{7.13}$$

is an integer ($p = 2, 3, \ldots$). Otherwise

$$m_{[p]} < \infty, \qquad m_{[p]+1} = \infty \tag{7.14}$$

Proof. We use Lemma 6.5, where $x = |\mathbf{k}|$, $a(s)$ and $b(s)$ are given in Eq. (4.11). Equation (7.13) is the corresponding particular case of Eq. (6.13). Hence,

$$(F, e^{-i\mathbf{k}\cdot\mathbf{v}}) = 1 - \frac{x^2}{2} + \sum_{l=2}^{[p]} a_l x^{2l} + bx^{2p} + \cdots, \qquad \mathbf{k} \to 0, \tag{7.15}$$

Then we use the well connection between moments of F and derivatives of its characteristic function:

$$[\Delta_{\mathbf{k}}^l (F, e^{-i\mathbf{k}\cdot\mathbf{v}})]_{\mathbf{k}=0} = (-1)^l\, (F, |\mathbf{v}|^{2l}) \qquad l = 1, 2, \ldots, \tag{7.16}$$

where $\Delta_{\mathbf{k}}$ is the Laplacian in $\mathbb{R}^n$. If $p \geqslant 2$ is an integer, then the expansion (7.15) does not contain fractional powers of $|\mathbf{k}|^2$ and therefore all moments of F are finite. If $p \geqslant 3/2$ is not an integer, then we need to prove that

$b \neq 0$ in Eq. (7.15). We prove this by contradiction. If $p \neq [p]$ and $b = 0$, then we proceed to the construction of an asymptotic (as $|\mathbf{k}|^2 \to 0$) expansion described in Lemma 6.5; we can continue the process indefinitely and no fractional powers of $|\mathbf{k}|^2$ will appear in the expansion if $b = 0$. Hence, the function $F(|\mathbf{v}|)$ has finite moments of all orders. However, it was already proved in ref. 7 (for the case $n = 3$, $p \neq [p]$; the proof can be generalized to an arbitrary $n = 2, 4, 5,\ldots$ without changes) that such function $F(|\mathbf{v}|)$ cannot be positive. This leads to a contradiction with the positivity of $F(|\mathbf{v}|)$ (Theorem 7.1). ∎

It should be stressed that the expansion (7.15) apparently first appeared in papers[3,4] with the formal conclusion, that

$$F(\mathbf{v}) \cong |\mathbf{v}|^{-(n+2p)} \qquad \mathbf{v} \to \infty, \tag{7.17}$$

This conclusion seems quite probable, but it is difficult to prove it rigorously and we did not try to do it.

All the results of this section are proved for the multidimensional case $n \geqslant 2$. For completeness, we present also similar results in the simplest case, $n = 1$. In such a case, the IBE is also defined by Eq. (7.1) with $n = 1$; however, Eqs. (7.2) and (7.3) should be replaced by

$$\phi(|k|, t) = \int_{-\infty}^{\infty} f(|v|, t)\, e^{-ikv}\, d\mathbf{v}, \qquad k \in \mathbb{R}, \tag{7.18}$$

and by Eq. (2.23) for $\phi(|k|, t)$, respectively.

We note that Eq. (2.23) coincides with Eq. (4.8), where $G(s) = \delta(s - z)$, $x = |k|$, $a(s) = s$, $b(s) = 1 - s$. The self-similar solution (Theorem 5.3) is given in Eqs. (2.24) and (2.25) in an explicit form. Theorem 6.2 (self-similar asymptotics for Eq. (4.8)) remains valid in this case. Hence, we proved the following

Lemma 7.3. Theorem 7.1 remains valid also in the case $n = 1$ for IBE (7.1) defined through Eqs. (7.18) and (2.23) as indicated above. The function $F(|\mathbf{v}|)$ and the parameter μ for $n = 1$ are given in an explicit form in Eqs. (2.24) and (2.25).

8. POWER-LIKE TAILS, THE ERNST-BRITO CONJECTURE, AND OPEN QUESTIONS

The main result of Section 7 is the proof of self-similar asymptotics (Ernst–Brito conjecture).[3,4] We did not prove it for arbitrary initial data, but at least Theorem 7.1 proves it for the most typical initial data (like a Maxwellian) and leaves almost no doubt that the conjecture is true for a wide class of initial conditions. On the other hand, the asymptotic equality

$$f(|\mathbf{v}|, t) \cong e^{n\mu t} F(|\mathbf{v}|\, e^{\mu t}), \qquad t \to \infty, \quad |\mathbf{v}| \to 0,$$

has, strictly speaking no relation to high energy tails of $f(|\mathbf{v}|, t)$; rather it actually describes a large time asymptotics for *small* energies, which may be large compared to $T(t) \to 0$. Therefore the power-like behavior of $F(|\mathbf{v}|)$ for large $|\mathbf{v}|$ does not seem to be very important. Moreover, this property is very unstable in some sense near the elastic limit, as one can see from the following

Lemma 8.1. There exists a sequence

$$\{e_m, m = 1, 2, \ldots\}, \qquad 0 \leqslant e_m < 1, \quad e_m \to 1$$

of values of the restitution coefficient e such that all the positive roots

$$p_m = p(z_m), \qquad m = 1, 2, \ldots, \quad z_m = \tfrac{1}{2}(1 + e_m)$$

of Eq. (7.13) are integers.

Proof. We denote the left hand side of Eq. (7.13) by $A(p, z)$. Then, for any fixed $p > 1$, $A(p, z)$ is a continuous function of $z \in [1/2, 1]$ and

$$A(p, 1) = \int_0^1 ds\, G(s)[1 - s^p - (1-s)^p] > 0,$$

$$A(p, 1/2) = 1 - \frac{p}{2} \int_0^1 ds\, sG(s) < 1 - \frac{p}{2},$$

Hence, $A(p, 1/2) < 0$ for sufficiently large p. Therefore for such p there exists a root $z = z_p \in (1/2, 1)$ of the equation $A(p, z) = 0$. Taking a sequence $p_m = N + m$, where N is a sufficiently large integer, we obtain a sequence $\{e_m\}$ given by the equalities

$$e_m = 2z_{N+m} - 1, \qquad m = 1, 2, \ldots$$

It remains to prove that $e_m \to 1$ or, equivalently, $z_{N+m} \to 1$, if $m \to \infty$. We omit some simple calculations showing that the asymptotic behavior of the root z_p of the equation $A(p, z_p) = 0$ is given by the equality

$$z_p = 1 - \frac{1}{p}\left[2 \int_0^1 ds\, sG(s)\right], \qquad p \to \infty,$$

and therefore $z_p \to 1$ if $p \to \infty$. ∎

Lemma 8.1 combined with Theorem 7.2 shows that, in the "near elastic case," $0<1-e<<1$, there exists a countable set of values of e such that the function $F(|\mathbf{v}|)$ has, for any such value of e, finite moments of all orders. This is a main difference of the many-dimensional case ($n\geqslant 2$) from the 1d case (see Eq. (2.22)) when $F(|\mathbf{v}|)$ (2.25) does not depend on the restitution coefficient (a new interpretation of the well-known universality of the BKW solutions satisfying EBE with an arbitrary kernel).

Thus the power-like tails of $F(|\mathbf{v}|)$ are not necessary for the Ernst–Brito conjecture to hold: the self-similar asymptotics (Theorem 7.2) holds independently of whether or not $F(|\mathbf{v}|)$ has such tails. Paradoxically, the logical basis for the Ernst–Brito conjecture (higher moments of the "rescaled" solution tend to infinity and possible existence of self-similar solutions with "the same" number of finite moments) is not, generally speaking, correct (the second argument fails if $F(|\mathbf{v}|)$ has finite moments of all orders). However the conjecture itself

$$f(|\mathbf{v}|,t)\cong e^{n\mu t}F(|\mathbf{v}|\,e^{\mu t}),\qquad t\to\infty,\quad |\mathbf{v}|\to 0, \tag{8.1}$$

(8.1)) appears to be true in the most general case of arbitrary kernels $G(s)$ satisfying the usual restrictions.

Remark 8.2. The following simple example

$$f(|\mathbf{v}|,t)=(1-e^{-nt})\,f_1(|\mathbf{v}|)+e^{-(n+3)\,t}f_2(|\mathbf{v}|e^{-t}),\qquad \mathbf{v}\in\mathbb{R}^3,\quad n=1,2,3,\ldots,$$

where $f_{1,2}(|\mathbf{v}|)$ are normalized distribution functions with finite moments of any order, shows that the divergence (for $t\to\infty$) of higher moments (of order greater than n) is not in contradiction with the fact that $f(|\mathbf{v}|,t)\to f_1(|\mathbf{v}|)$ as $t\to\infty$ (point-wise if $f_2(|\mathbf{v}|)$ is bounded and in the sense of probability measures in the most general case).

We mention two open questions:

(a) How can we describe the whole class of initial conditions satisfying the Ernst–Brito conjecture? The proof of Theorem 7.1 can be easily generalized to anisotropic solutions of IBE. On the other hand, the second assumption in part (3) of the theorem (positivity of $\mathscr{F}[f_0]$) is probably not necessary, but our proof is based on it.

(b) How can we connect the self-similar solutions of the Fourier transformed EBE (Sections 5–6) with its "physical" (positive) solutions? This question is partly addressed in Section 9; we also discuss some more general aspects of it below.

In order to explain the second question, we mention that Theorem 5.3 means, in particular, that, for any

$$0 < \mu < \mu_{\max} = \int_0^1 ds\, G(s)\, s(1-s)$$

there exists a unique self-similar solution $\phi_\mu(x, t) = \psi_\mu(xe^{-\mu t})$ such that

$$0 \leqslant \psi_\mu(x) \leqslant 1; \qquad \phi_\mu(x) = 1 - \tfrac{1}{2}\, x^p + \cdots, \qquad x \to 0,$$

where $1 < p \leqslant 2$ is defined as the smallest root of the equation

$$\mu = \frac{1}{p} \int_0^1 ds\, G(s)[1 - s^p - (1-s)^p].$$

This solution for $p = 2$ is known explicitly from Eq. (2.24) with $\mu = \mu_{\max}$.

Thus Theorem 5.3 leads to the following

Proposition 8.3. The (elastic) Boltzmann equation for Maxwell molecules ($n = 3$) admits a class of self-similar solutions

$$\begin{gathered} f_\mu(|\mathbf{v}|, t) = e^{3\mu t/2} F(|\mathbf{v}|\, e^{\mu t/2}), \qquad 0 < \mu \leqslant \mu_{\max}, \\ F(|\mathbf{v}|) = \int_{\mathbb{R}^3} \psi_\mu\left(\frac{|\mathbf{k}|^2}{2}\right) e^{i\mathbf{k}\cdot\mathbf{v}}\, \frac{d\mathbf{v}}{(2\pi)^3}, \end{gathered} \tag{8.2}$$

having zero second moment. This leads to a class of solutions with finite (and positive) second moment given by the convolution

$$f_{\mu,\theta}(|\mathbf{v}|, t) = f_\mu(|\mathbf{v}|, t) * M_\theta(|\mathbf{v}|), \qquad M_\theta(|\mathbf{v}|) = (2\pi\theta)^{-3/2}\, e^{-|\mathbf{v}|^2/(2\theta)}. \tag{8.3}$$

The proof for true Maxwell molecules can be easily given by successive approximation by cutoff models (as in our previous paper[(2)]). The possibility of performing the inverse Fourier transform is justified by the inequality (5.12).

We recall that the analogous statement (with $\mu \in (-\infty, 0)$) was previously proved[(2)] for solutions with infinite energy; moreover all such solutions appeared to be non-negative. The case of finite energy is obviously more complex. The solutions $f_\mu(|\mathbf{v}|, t)$ (8.2) cannot be positive since they have a vanishing second order moment. The "convoluted" solutions (8.3) might be in principle be positive for a sufficiently large $\theta > 0$, but we were not able neither to prove nor disprove it. The simplest example (2.24) does not lead to non-negative solutions of the Boltzmann equation, as we already remarked in ref. 2. On the other hand, the solutions (8.2) do play a certain asymptotic role as follows from Theorem 6.2.

Finally we mention that the solutions (8.2), (8.3) represent new eternal solutions (in addition to ref. 2) of the Boltzmann equation (i.e., solutions existing for all $t \in (-\infty, \infty)$). Unfortunately we cannot say much on their positivity.

9. MAXWELLIAN TAILS FOR EBE

It was noted in Section 8 that the class of self-similar solutions of IBE can be used to study the large time asymptotic behavior for small energies. On the contrary, we shall see below that Theorem 6.2 does bring some new information about high energy tails in the classical (elastic) case.

Let $f(\mathbf{v}, t)$ ($\mathbf{v} \in \mathbb{R}^n, t \in \mathbb{R}_+$) be a solution of the Elastic Boltzmann Equation (EBE)

$$\frac{\partial f}{\partial t} = Q(f.f), \qquad f_{|t=0} = f_0(\mathbf{v}) \geqslant 0. \tag{9.1}$$

We assume that

$$(f_0, 1) = 1, \qquad \frac{1}{n}(f_0, |\mathbf{v}|^2) = 1, \qquad \frac{1}{n}(f_0, \mathbf{v}) = 0 \tag{9.2}$$

in the notation (2.5). In addition we assume that $f_0(\mathbf{v})$ is a rapidly decreasing function:

$$(f_0, e^{\alpha_0 |\mathbf{v}|^2}) < \infty \tag{9.3}$$

for some $\alpha_0 > 0$. Then we can use the following

Lemma 9.1. There exists a solution $f(\mathbf{v}, t)$ of the problem (9.1) such that the equalities (9.2) and the inequality (9.3) (perhaps with another constant $0 \leqslant \alpha_1 < \alpha_0$) are valid for all $t > 0$. The function

$$\Phi(\mathbf{k}, t) = \int_{\mathbb{R}^n} \frac{d\mathbf{v}}{(2\pi)^{n/2}} e^{-|\mathbf{v}-\mathbf{k}|^2/2} F(\mathbf{v}, t), \qquad f(\mathbf{v}, t) = F(\mathbf{v}, t) \frac{e^{-|\mathbf{v}|^2/2}}{(2\pi)^{n/2}} \tag{9.4}$$

satisfies the equation

$$\frac{\partial \Phi}{\partial t} = \int_{S^{n-1}} d\boldsymbol{\omega}\, g\left(\frac{\mathbf{k}\cdot\boldsymbol{\omega}}{|\mathbf{k}|}\right)\left[\Phi\left(\frac{\mathbf{k}+|\mathbf{k}|\,\boldsymbol{\omega}}{2}\right)\Phi\left(\frac{\mathbf{k}-|\mathbf{k}|\,\boldsymbol{\omega}}{2}\right)-\Phi(\mathbf{k})\right], \tag{9.5}$$

and conditions

$$\Phi_{|\mathbf{k}=0} = 1, \qquad \nabla\Phi_{|\mathbf{k}=0} = 0, \qquad \Delta\Phi_{|\mathbf{k}=0} = 0 \tag{9.6}$$

Proof. The fact that there exists a global solution of the problem (9.1) in the class of rapidly decreasing functions (9.3) is well-known (see ref. 14; the same property for hard spheres is proved in ref. 20). The Fourier transform $\phi(\mathbf{k}) = (f, e^{-i\mathbf{k}\cdot\mathbf{v}})$ for such functions can be extended to all complex values $\mathbf{k} \in \mathbb{C}^n$, in particular to imaginary values $\mathbf{k} = i\mathbf{q}$, $\mathbf{q} \in \mathbb{R}^n$. The usual equation for $\phi(\mathbf{k}, t)$ remains the same for all complex $\mathbf{k}$ and remains invariant under multiplication by $\exp(\beta\, |\mathbf{k}|^2)$ with any (complex) $\beta = \text{const}$. The transformation (9.4) reads

$$\Phi(\mathbf{k}, t) = e^{-|\mathbf{k}|^2/2}(f, e^{\mathbf{k}\cdot\mathbf{v}})$$

and therefore leads to the familiar equation (9.5). Equations (9.6) are simply obvious consequences of the conservation laws and the normalization conditions (9.2). ▮

The transformation (9.4) leads to the following idea to study the asymptotic properties of $f(\mathbf{v}, t)$, as $|\mathbf{v}| \to \infty$. Let us consider any function $\epsilon(t)$ such that

$$\epsilon(t) > 0; \qquad \epsilon(t) \to 0, \qquad t \to \infty.$$

Then

$$\Phi\left(\frac{\mathbf{k}}{\epsilon(t)}, t\right) = \int_{\mathbb{R}^n} d\mathbf{v}\, \Gamma(\mathbf{k} - \mathbf{v}; \epsilon(t))\, F\left(\frac{\mathbf{v}}{\epsilon(t)}, t\right), \tag{9.7}$$

where

$$\Gamma(\mathbf{k}; \epsilon) = (2\pi\epsilon^2)^{-n/2} \exp\left(-\frac{|\mathbf{k}|^2}{2\epsilon^2}\right) \tag{9.8}$$

Noting that $\Gamma(\mathbf{k}; \epsilon) \to \delta(\mathbf{k})$, as $\epsilon \to 0$, we formally obtain an asymptotic equality

$$\Phi\left(\frac{\mathbf{k}}{\epsilon(t)}, t\right) \cong F\left(\frac{\mathbf{v}}{\epsilon(t)}, t\right), \qquad t \to \infty \tag{9.9}$$

which can be proved rigorously in many important cases. We remark that, for rapidly decreasing solutions $f(|\mathbf{v}|, t)$ of the Boltzmann equation (9.1), the function $\Phi(\mathbf{k}, t)$ must be an entire analytic function of the (vector) complex variable $\mathbf{k} \in \mathbb{C}^n$. Hence, the only class of the self-similar solutions, which have been discussed above (see the end of Section 8) and can be considered in the present context, is the set of BKW solutions:

$$\Phi_*(\mathbf{k}, t) = \psi_{\mathrm{BKW}}\left(a\,\frac{|\mathbf{k}|^2}{2}\,e^{-\mu t}\right) \tag{9.10}$$

in the notation of (2.24) with an arbitrary parameter $a > 0$. It was already proved (Theorem 6.2) that any isotropic solution $\Phi(|\mathbf{k}|, t)$ such that

$$\begin{aligned} &\Phi(|\mathbf{k}|, 0) = \phi_0\left(a\,\frac{|\mathbf{k}|^2}{2}\right), \qquad 0 \leqslant \phi_0(x) \leqslant 1, \\ &\phi_0(x) = 1 - \frac{1}{2}\,x^2 + O(x^{2+\delta}), \qquad x \to 0, \quad \delta > 0. \end{aligned} \tag{9.11}$$

has the self-similar asymptotics

$$\lim_{t \to \infty} \Phi(|\mathbf{k}|\,e^{\mu t}, t) \to \psi_{\mathrm{BKW}}\left(a\,\frac{|\mathbf{k}|^2}{2}\right), \tag{9.12}$$

and

$$0 \leqslant \Phi(|\mathbf{k}|\,e^{\mu t}, t) \leqslant 1, \tag{9.13}$$

Let us assume that there exists a $\delta > 0$ such that

$$\Phi(|\mathbf{k}|, 0) \leqslant \psi_{\mathrm{BKW}}\left(\delta\,\frac{|\mathbf{k}|^2}{2}\right), \tag{9.14}$$

then

$$\Phi(|\mathbf{k}|\,e^{\mu t}, t) \leqslant \psi_{\mathrm{BKW}}\left(\delta\,\frac{|\mathbf{k}|^2}{2}\right),$$

for all $t \geqslant 0$. The transformation (9.4) implies that

$$(\Phi, 1) = (F, 1);$$

moreover both functions Φ and F are non-negative. Let us consider any function $\Phi(|\mathbf{k}|, t)$ satisfying (9.5) and (9.6) and the corresponding function $F(|\mathbf{v}|, t)$. Denoting

$$\tilde{F}(|\mathbf{v}|, t) = F(|\mathbf{v}|\,e^{\mu t}, t), \qquad \tilde{\Phi}(|\mathbf{k}|, t) = \Phi(|\mathbf{k}|\,e^{\mu t}, t),$$

we apply (9.7) with $\epsilon = e^{-\mu t}$ and obtain

$$\tilde{\Phi}(|\mathbf{k}|, t) = \int_{\mathbb{R}^n} d\mathbf{v}\, \Gamma(\mathbf{k}-\mathbf{v}; e^{-\mu t})\, \tilde{F}(|\mathbf{v}|, t),$$

By usual arguments[15] (Chapter 8), we can conclude that in such a case the limiting equality (9.12) implies the weak convergence (in the sense of measures):

$$\tilde{F}(|\mathbf{v}|, t) \to \psi_{\mathrm{BKW}}\left(a \frac{|\mathbf{v}|^2}{2}\right).$$

The condition (9.14) guarantees that the integral $(\tilde{\Phi}, 1)$ is uniformly bounded for all $t > 0$ and this justifies passing to the limit. The result can be formulated as follows

Theorem 9.2. (1) There exists a non-empty class of initial conditions $f_0(|\mathbf{v}|)$ satisfying (9.2) and (9.3) such that the corresponding solutions $f(|\mathbf{v}|, t)$ of problem (9.1) have the following asymptotic behavior for large $|\mathbf{v}|$ and large t:

$$f(|\mathbf{v}|, t) = F(|\mathbf{v}|, t)\, \frac{e^{-|\mathbf{v}|^2/2}}{(2\pi)^{n/2}}, \tag{9.15}$$

$$F(|\mathbf{v}|\, e^{\mu t}, t) \to_{t\to\infty} (1 + a|\mathbf{v}|^2/2)\, e^{-a\,|\mathbf{v}|^2/2}, \qquad \mu = \int_0^1 ds\, G(s)\, s(1-s), \tag{9.16}$$

where the convergence is understood in the sense of measures and the value of the parameter a depends on the initial conditions in the following way:

$$a = \sqrt{1 - \frac{m_4}{n(n+2)}}, \qquad m_4 = (f_0, |\mathbf{v}|^4).$$

(2) Any function $f_0(|\mathbf{v}|)$ satisfying conditions (9.2), (9.3) and the assumption (9.14) with some $\delta > 0$ for the corresponding $\Phi_0(|\mathbf{k}|)$ belongs to this class. In particular in the three-dimensional case all functions having finite support such that

$$f_0(|\mathbf{v}|) \geqslant 0, \qquad f_0(|\mathbf{v}|) = 0 \quad \text{if} \quad |\mathbf{v}| > r_0, \quad r_0 < \sqrt{5}, \tag{9.17}$$

and satisfying Eqs. (9.2) and (9.3) belong to this class.

Proof. It is easy to verify that the above explicit formula for a leads to the correct asymptotics (9.11) (as $|\mathbf{k}|\to 0$) of the function $\Phi_0(|\mathbf{k}|)=\Phi(|\mathbf{k}|,0)$. Accordingly, we omit the direct calculations. The proof of the asymptotic formula (9.16) under the assumption (9.14) was already given above. The We need only to verify that the assumption (9.14) is fulfilled for the functions (9.17) with compact support. The transformation (9.4) leads to

$$\begin{aligned} e^{|\mathbf{k}|^2/2}\Phi_0(\mathbf{k}) &= \frac{4\pi}{|\mathbf{k}|}\int_0^\infty dr\, r f_0(r)\sinh(|\mathbf{k}|r) \\ &\leqslant 1+\sum_{n=01}^{\infty}\frac{x^n}{n!}\frac{3r_0^{2(n-1)}}{(2n+1)!!}, \qquad x=|\mathbf{k}|^2/2. \end{aligned} \tag{9.18}$$

The condition in (9.14) is satisfied if

$$e^{|\mathbf{k}|^2/2}\Phi_0(\mathbf{k})\leqslant(1+\delta x)\,e^{-(1-\delta)\,x}=1+\sum_{n=1}^{\infty}\frac{x^n}{n!}(1-\delta)^{n-1}\,[1+(n-1)\,\delta] \tag{9.19}$$

for some $\delta>0$ and any $x>0$. It is sufficient to show that there exists a $\delta>0$ such that

$$\frac{3r_0^{2(n-1)}}{(2n+1)!!}\leqslant(1-\delta)^{n-1}\,[1+(n-1)\,\delta], \qquad n=1,2,..., \quad r_0^2<5 \tag{9.20}$$

This result will be a consequence in the lemma below, by letting there $\theta=\delta=(1-r_0^2/5)^{1/2}$. Therefore the estimate in (9.19) is proved and the function (9.17) does satisfy the conditions in (9.14). ∎

In order to complete the proof of Theorem 9.2 let us prove the following

Lemma 9.3. The following inequalities

$$\frac{3[5(1-\theta^2)]^{n-1}}{(2n+1)!!}\leqslant(1-\theta)^{n-1}\,[1+(n-1)\,\theta], \qquad n=1,2,... \tag{9.21}$$

hold for any $0\leqslant\theta\leqslant 1$.

Proof. We need to prove that

$$r_n=\frac{3[5(1+\theta)]^{n-1}}{(2n+1)!!\,[1+(n-1)\,\theta]}\leqslant 1, \qquad 0\leqslant\theta\leqslant 1, \quad n=1,2,...$$

Noting that

$$r_1 = 1, \qquad r_{n+1} = \alpha_n(\theta)\, r_n, \qquad \alpha_n(\theta) = \frac{5(1+\theta)[1+(n-1)\,\theta]}{(2n+3)(1+n\theta)},$$

we verify that $\alpha_n'(\theta) \geqslant 0$ for any $\theta \geqslant 0$, $n \geqslant 1$. Hence,

$$\alpha_n(\theta) \leqslant \alpha_n(1) = \frac{10n}{(2n+3)(1+n)} \leqslant 1, \qquad n = 1, 2,\ldots,$$

and the lemma is proved. ∎

Remark 9.4. Part (1) of Theorem 9.2 is in a sense obvious since the known exact solution

$$f_{\mathrm{BKW}}(|\mathbf{v}|, t) = [2\pi(1-a(t))]^{n/2}\left[1+\frac{a(t)}{1-a(t)}\left(\frac{|\mathbf{v}|^2}{1-a(t)}-\frac{n}{2}\right)\right] e^{-\frac{|\mathbf{v}|^2}{2(1-a(t))}},$$

$$a(t) = ae^{-2\mu t}, \qquad |\mathbf{v} \in \mathbb{R}^n$$

which is positive for all $t>0$ if $a \leqslant 2/(n+2)$, certainly belongs to the described class. Part (2) shows that this class includes also other solutions having the same asymptotic behavior, in particular the functions satisfying conditions (9.17).

Another interesting example is given by the initial conditions

$$f_{|t=0} = f_p(|\mathbf{v}|) = A|\mathbf{v}|^{2p}\, e^{-\alpha \frac{|\mathbf{v}|^2}{2}}, \qquad p = 1, 2,\ldots, \quad \mathbf{v} \in \mathbb{R}^3$$

where A and α are chosen in such a way that Eqs. (9.2) (with $n = 3$) hold. In rder to prove that the inequality (9.14) is also valid in this case, we note that

$$\mathscr{F}[f_p(|\mathbf{v}|)] = \int_{\mathbb{R}^3} f_p(\mathbf{v})\, e^{-i\mathbf{k}\cdot\mathbf{v}}\, d\mathbf{v} = e^{-\frac{|\mathbf{k}|^2}{2\alpha}} \frac{L_p^{1/2}(|\mathbf{k}|^2/2\alpha)}{L_p^{1/2}(0)}, \qquad p = 1, 2,\ldots,$$

where $L_p^{1/2}(z)$ denotes a Laguerre polynomial. It is well-known that the zeroes of $L_p^{1/2}(z)$ are real and positive. Hence

$$L_p^{1/2}(z) = L_p^{1/2}(0) \prod_{k=1}^{p} (1-\beta_k^{(p)} z), \qquad \beta_k^{(p)} > 0.$$

Therefore the function $\Phi_p(|\mathbf{k}|) = \Phi(|\mathbf{k}|, 0)$ (9.4) reads

$$\Phi(|\mathbf{k}|, 0) = \phi_0(|\mathbf{k}|^2/2\alpha), \qquad \phi_0(x) = \prod_{k=1}^{p} (1+\beta_k^{(p)} x)\, e^{-\beta_k^{(p)} x},$$

and the condition (9.14) where δ equals any $\beta_k^{(p)}$ should be fulfilled. The same argument can be used for any rapidly decreasing isotropic initial function (9.3) such that its Fourier transform has just real zeroes ($z_n = |\mathbf{k}|_n^2$) (the condition (9.3) guarantees that $\mathscr{F}[f_0]$ is an entire analytic function of exponential type of the complex variable $z = |\mathbf{k}|^2$).

Remark 9.5. The condition (9.14) yields actually much more than merely a weak convergence of measures. It leads to convergence of all moments of $\tilde{F}(|\mathbf{v}|, t)$, etc. If this condition is replaced by a weaker one

$$0 \leqslant \Phi(|\mathbf{k}|, 0) \leqslant 1,$$

then it is still possible to prove the convergence (9.16) in the weak sense on continuous test functions with compact support.

It should be stressed, however, that not *all* rapidly decreasing initial conditions normalized as in Eq. (9.2) lead to the asymptotic behavior described by Eqs. (9.15). This is not true even for initial conditions with compact support. It was proved long ago (see ref. 16 for details) that there are two different kinds of asymptotic behavior for such initial conditions. The difference is expressed in terms of the so-called *tail temperature* $\tau(t)$ defined for any given solution $f(\mathbf{v}, t)$ of (9.1) by the following equality

$$\tau^{-1}(t) = \sup\{\alpha > 0 : (f, e^{\alpha |\mathbf{v}|^2/2}) < \infty\}$$

The function $\tau(t)$ is monotone increasing; moreover $\tau_M = 1$ for the Maxwellian normalized by (9.2). If $\tau(0) = 0$ (this holds, e.g., for initial data with compact support) then we distinguish between.

(a) the normal kind of asymptotic behavior if $\tau(t) \to 1$ as $t \to \infty$, and

(b) an abnormal kind of behavior if $\tau(t) \to \tau_\infty > 1$ as $t \to \infty$. It is known that τ_∞ can be as large as we want for initial conditions with $\tau(0) = 0$ and with the usual normalization (9.2).

Theorem 9.2 obviously describes a normal kind of asymptotics. It seems quite probable that all solutions of that kind with $\tau(0) = 0$ belong to the class described by Theorem 9.2, but we were not able to prove this. The problem of the asymptotic behavior of the solutions belonging to the second (abnormal) kind remains open.

10. CONCLUDING REMARKS

We have considered some questions related to the self-similar asymptotics in the theory of the Boltzmann equation for both elastic (EBE) and inelastic particles (IBE). In the second case we think of granular materials, when the model of hard spheres with inelastic collisions is replaced by a Maxwell model, characterized by a collision frequency independent of the relative speed of the colliding particles (with a corresponding change of the time variable).

We stress an important difference between Maxwell models in the two cases (elastic and inelastic). In the first case the Boltzmann equation is clearly related to the classical dynamics of the gas molecules, which are, by definition, mass points interacting with a potential $U(r)$ proportional the inverse fourth power of their distance. It is true that we often use the cut-off cross section (pseudo-Maxwell molecules), but this is simply a mathematically well-defined approximation of the EBE for true Maxwell molecules: almost all results can be (quite rigorously) generalized to the true Maxwell cross-section by the corresponding transition to the limit.

On the other hand, in the inelastic case we should consider a kinetic equation which corresponds to the dynamics of macroscopic particles of finite size (say, inelastic hard spheres). This obviously means that the basic equation (in the low density limit) is the IBE for hard spheres. Hence, the inelastic Maxwell model does not have an underlying particle dynamics: it is just a mathematical model though with a clear probabilistic meaning (therefore it is very easy to make numerical experiments with this model[5]). For this reason, we need to understand clearly the relation between the Maxwell model and the "true" IBE for hard spheres and we tried to clarify this question (in which sense the model can be considered as an "approximation" of the IBE for hard spheres) in Section 2.

One should be very careful when trying to draw any conclusion for the hard sphere case on the basis of Maxwell models. There are many examples (see, e.g., ref. 24) showing that some properties of EBE for Maxwell molecules cannot be generalized to the case of hard spheres.

At least, the Boltzmann equation for both elastic and inelastic Maxwell particles is a well-defined mathematical model. Fortunately, we already have a mature technique (based on the Fourier transform) to deal with EBE, which can be easily generalized to the case of IBE. Moreover, it is natural to consider both cases from a unified viewpoint, as we do in this paper. Then mathematical methods, developed for EBE, allow us (and this does not occur very often in kinetic theory) to answer some relevant questions concerning the self-similar asymptotics in a quite rigorous form. These answers are given above in Theorems 7.1 and 9.2.

The main results can be easily explained to a reader who is not interested in mathematical details and proofs. For simplicity, we shall do this for the case $n = 3$.

1. What new results do we obtain for IBE?

We consider the initial value problem (7.1), where $f_0(|\mathbf{v}|)$ is normalized as

$$(f, 1) = 1, \qquad (f, |\mathbf{v}|^2) = 3. \tag{10.1}$$

It is well-known that

$$f(|\mathbf{v}|, t) \to \delta(|\mathbf{v}|), \qquad T(t) = \tfrac{1}{3}(f, |\mathbf{v}|^2) = e^{-2\mu t}, \qquad \mu = \tfrac{1}{8}(1 - e^2), \tag{10.2}$$

where $0 \leqslant e < 1$ is the restitution coefficient (note that $G(s) = 1$ in Eq. (7.7) for $n = 3$).

We have constructed (Theorem 7.1) the positive self-similar solution of Eq. (7.1)

$$f_S(|\mathbf{v}|, t) = e^{3\mu t} F(|\mathbf{v}|\, e^{\mu t}), \tag{10.3}$$

and proved that for any initial condition $f_0(\mathbf{v})$ satisfying Eqs. (10.1) and two additional assumptions

$$(f, |\mathbf{v}|^{2+\epsilon}) < \infty, \qquad (f, e^{-i\mathbf{k}\cdot\mathbf{v}}) \geqslant 0, \quad \mathbf{k} \in \mathbb{R}^3, \tag{10.4}$$

the corresponding solution $f(|\mathbf{v}|, t)$ of the problem (7.1) has a self-similar asymptotics:

$$f(|\mathbf{v}|, t) \cong F_S(|\mathbf{v}|, t), \qquad \text{if} \quad t \to \infty, \quad |\mathbf{v}| \to 0, \tag{10.5}$$

such that $|\mathbf{v}|\, e^{\mu t}$ remains finite. The exact meaning of this asymptotic equality is the following

$$e^{-3\mu t} f(|\mathbf{v}|\, e^{-\mu t}, t) \xrightarrow[t \to \infty]{} F(|\mathbf{v}|), \tag{10.6}$$

where the convergence is understood in the sense of probability measures.

Thus, we have proved the Ernst–Brito conjecture for a wide class of initial data, including the Maxwellian

$$M(|\mathbf{v}|) = (2\pi)^{-3/2}\, e^{-|\mathbf{v}|^2/2}, \tag{10.7}$$

all satisfying the conditions (10.4).It is also shown that the function $F(|\mathbf{v}|)$ has a restricted number of moments not for all, but for almost all values of the restitution coefficient $0 \leqslant e < 1$; there exists a sequence $\{e_n\}$, $e_n \to 1$ such that the function $F(|\mathbf{v}|)$ has finite moments of all orders if $e = e_n$, $n = 1, 2,\ldots$ (this degeneracy is discussed in detail at the beginning of Section 8).

2. What new results do we obtain for EBE?

We consider the initial value problem (9.1) for isotropic initial data $f_0(|\mathbf{v}|) \geqslant 0$ satisfying Eqs. (10.1). It is well known that $f(|\mathbf{v}|, t) \to_{t\to\infty} M(|\mathbf{v}|)$, $M(|\mathbf{v}|)$ is given by Eq. (10.7)

In addition, we assume that

$$(f_0, e^{\mathbf{k}\cdot\mathbf{v}}) \leqslant (M, e^{\mathbf{k}\cdot\mathbf{v}}), \qquad \mathbf{k} \in \mathbb{R}^3 \tag{10.8}$$

and study the asymptotic properties of the function $g(|\mathbf{v}|, t) = f(|\mathbf{v}|, t)/M(|\mathbf{v}|)$ for large values of $|\mathbf{v})$ and t. This can be understood as the problem of the formation of the Maxwellian tails for initial data satisfying (10.8). It is proved (Theorem 9.2) that $g(|\mathbf{v}|, t)$ has a self-similar asymptotics

$$g(|\mathbf{v}|\, e^{\mu t}, t) \xrightarrow[t\to\infty]{} (1+a|\mathbf{v}|^2)\, e^{-a|\mathbf{v}|^2}, \tag{10.9}$$

where $a > 0$ depends only on $f_0(|\mathbf{v}|)$, μ depends only on the cross-section (see the explicit formulas in Theorem 9.2), and the convergence is understood in a certain weak sense. We note that the above asymptotic formula obviously holds for the exact BKW solution. Krook and Wu,[22] 26 years ago, conjectured that this exact solution describes in some sense a large time asymptotics of solutions $f(|\mathbf{v}|, t)$. The conjecture was never confirmed in its initial form (see ref. 23 for a review). Our results for EBE can be considered as a proof of a modified form (10.9) of the Krook–Wu conjecture. The restriction (10.8) seems to be not only sufficient but also necessary for self-similar asymptotics in the form (10.8). Examples of initial conditions leading to Eq. (10.9) are

(1) functions with compact support

$$f_0(|\mathbf{v}|) \geqslant 0, \qquad f_0(|\mathbf{v}|) = 0 \qquad \text{if} \quad |\mathbf{v}| > \sqrt{5};$$

(2) functions of the form

$$f_0(|\mathbf{v}|) = \text{const.}\, |\mathbf{v}|^{2p}\, e^{-\alpha|\mathbf{v}|^2}, \qquad p = 1, 2,...$$

The two types of self-similar asymptotics given by Eq. (10.6) for IBE and by Eq. (10.9) for EBE seem to be very different from the physical viewpoint. Both of them, however, are direct consequences of the self-similar asymptotics for solutions of Eq. (4.9) (Theorem 6.2). Therefore it is quite natural, from a mathematical viewpoint, to present them together in this paper, as particular cases of a general non-linear kinetic equation. These results are, to our knowledge, the first rigorous results on self-similar asymptotics for finite energy solutions of nonlinear Boltzmann equations. (similar questions for solutions with infinite energy were discussed in our previous paper[2]).

ACKNOWLEDGMENTS

The research described in the paper was performed in the frame of European TMR (contract n. ERBFMRXCT97O157) and was also partially supported by MURST of Italy and NFR of Sweden. The idea and the first details were elaborated during a joint visit of the authors at Institute Henri Poincaré, Paris (France) in the frame of a trimester program on "*Limites Hydrodynamiques*," organized by F. Golse and S. Olla. The authors are grateful for the invitation and the hospitality. The authors are also grateful to M. Ernst, for sending them the preprint version of refs. 3 and 4. Finally the authors thank L. Arkeryd and V. Kolyada for an important discussion on the content of Remark 9.5.

REFERENCES

1. A. V. Bobylev and C. Cercignani, Exact eternal solutions of the Boltzmann equation, *J. Statist. Phys.* **106**:1019–1038 (2002).
2. A. V. Bobylev and C. Cercignani, Self-similar solutions of the Boltzmann equation and their applications, *J. Statist. Phys.* **106**:1039–1071 (2002).
3. M. H. Ernst and R. Brito, High energy tails for inelastic Maxwell models, Preprint (submitted to *Europhys. Lett.*, 2001).
4. M. H. Ernst and R. Brito, Scaling solutions of inelastic Boltzmann equations with overpopulated high energy tails, Preprint (submitted to *J. Statist. Phys.*, 2002).
5. A. Baldassarri, U. Marini Bettolo Marconi, and A. Puglisi, Influence of correlations on the velocity statistics of scalar granular gases, *Europhys. Lett.* **58**:14–20 (2002).
6. P. L. Krapivsky and E. Ben-Naim, Nontrivial velocity distributions in inelastic gases, *J. Phys. A: Math. Gen.* **35**:L147–L152 (2002).
7. A. V. Bobylev, J. A. Carrillo, and I. A. Gamba, On some properties of kinetic and hydrodynamic equations for inelastic interaction, *J. Statist. Phys.* **98**:743–773 (2000).
8. C. Cercignani, J. A. Carrillo, and I. A. Gamba, Steady states of a Boltzmann equation for driven granular media, *Phys. Rev. E* **62A**:7700–7707 (2000).
9. C. Cercignani, Shear flow of a granular material, *J. Statist. Phys.* **102** (2001).
10. A. V. Bobylev and C. Cercignani, Moment equations for a granular material in a thermal bath, *J. Statist. Phys.* **106**:547–567 (2002).
11. C. Cercignani, *The Boltzmann Equation and its Applications* (Springer, New York, 1985).
12. E. Ben-Naim and P. Krapivsky, Multiscaling in inelastic collisions, *Phys. Rev. E* **61**:R5–R8 (2000).
13. A. V. Bobylev, J. A. Carrillo, and I. A. Gamba, Erratum on "On some properties of kinetic and hydrodynamic equations for inelastic interaction," *J. Statist. Phys.* **103**: 1137–1138 (2001)
14. A. V. Bobylev, The theory on the nonlinear spatially uniform Boltzmann equation for Maxwell molecules, *Sov. Sci. Rev. C Math. Phys.* **7**:111–233 (1988).

15. W. Feller, *An Introduction to Probability Theory and its Applications*, Vol. II (Wiley, New York, 1971).
16. A. V. Bobylev, On Fourier transform methods for the Boltzmann equation for Maxwell molecules, *Sov. Phys. Dokl.* **20**:820–822 (1976).
17. A. V. Bobylev, A class of invariant solutions of the Boltzmann equation, *Doklady Akad. Nauk SSSR* **231**:571–574 (1976).
18. A. V. Bobylev, Exact solutions of the Boltzmann equation, *Dokl. Akad. Nauk SSSR*, **225**:1296–1299 (1975).
19. T. Krook and T. T. Wu, Formation of Maxwellian tails, *Phys. Rev. Lett.* **36**:1107–1189.
20. A. V. Bobylev, Moment inequalities for the Boltzmann equation and applications to spatially homogeneous problems, *J. Statist. Phys.* **88**:1183–1214 (1997).
21. E. Lukacs, *Characteristic Functions* (Griffin, London, 1970).
22. M. Krook and T. T. Wu, Exact solutions of the Boltzmann equation, *Phys. Fl.* **20**:1589–1595 (1977).
23. M. Ernst, Nonlinear model Boltzmann equations and exact solutions, *Phys. Rep.* **78**:1–171 (1981).
24. A. V. Bobylev and C. Cercignani, Self-similar solutions of the Boltzmann equation for non-Maxwell molecules, to appear in *J. Statist. Phys.* (2002)